Theorem: First principle of Finite induction.

Let S be a set of positive integers with the following properties:

(a) The integer $1 \in S$

(b) whenever the integer $K \in S$, the next integer $K+1$ must also be in S.

Then S is the set of all positive integers, i.e. $S = \mathbb{N}$.

Pf. Let T be the set of all positive integers not in S, and assume that T is nonempty. ~~If $T = \phi$ then $S = \mathbb{N}$~~. By well ordering principle T will have a least element, say a. Since $1 \in S$, so $a > 1$. and so $0 < a-1 < a$. Since a is the smallest positive integer in T so $a-1 \notin T$. ~~$\Rightarrow$ This is~~ This implies $a-1 \in S$. By hypothesis (b), S must also contain $(a-1)+1 = a$ which is a contradiction to the fact that $a \in T$.

Hence we conclude that T is empty.

Hence $S = \mathbb{N}$.

Ex. Using principle of mathematical induction prove that

(a) $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

(b) $$1 + 3 + 5 + \cdots + (2n-1) = n^2 \quad \text{for all } n \geq 1.$$

Ex. Using Mathimatical induction prove that

$$1+3+5+\cdots+(2n-1) = n^2 \text{ for all } n \geq 1$$

soln.

$$1+3+5+\cdots+(2n-1) = n^2$$

for $n=1$, $LHS = 2\times1-1 = 1$

$RHS = 1^2 = 1$

$\therefore$ the formula is true for $n=1$.

Let the formula be true for $n=k$

$$\therefore 1+3+5+\cdots+(2k-1) = k^2$$

Now,

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)$$

$$= k^2+2k+2-1$$

$$= k^2+2k+1$$

$$= (k+1)^2$$

So the result is true for $n=k+1$.

Hence by Mathematical induction the result is true for all natural number n.

Scanned with CamScanner

Division Algorithm. Given integers a and b, with $b>0$, there exist unique integers q and r satisfying

$$a = qb + r, \quad 0 \le r < b$$

The integers q and r are called, respectively, the quotient and remainder in the division of a by b

Pf:- Consider the set $S = \{a - xb \mid x \text{ is an integer}, \ a - xb \geq 0\}$

Since $b>0$ so $b \geq 1 \Rightarrow |a|\, b \geq |a| \cdot 1$

$$\Rightarrow |a|\, b \geq |a|$$

Now,

$$a - (-|a|)\, b = a + |a|\, b \geq a + |a| \geq 0$$

Since a is an integer so $-|a|$ is also an integer.

Taking $x = -|a|$, we get

$$a - (-|a|)\, b = a - xb \geq 0$$

$\therefore$ The set S is nonempty. Thus S is a non empty set of non negative integers. So by well ordering principle S contains a smallest integer, call it r. By definition of S, there exists an integer q satisfying

$$r = a - qb, \quad \cancel{r \geq 0} \quad 0 \le r.$$

$$\Rightarrow a = qb + r, \quad 0 \le r.$$

We claim $r < b$. If this were not the case, then $\cancel{r \geq b}$ $r \geq b$ and

$$a - (q+1)\, b = (a - qb) - b = r - b \geq 0.$$

$$\Rightarrow a - (q+1)\, b \in S.$$

But $a - (q+1)\, b = r - b < r$ as $b > 0$. which is a contradiction because r is the smallest integer of S. Hence $r < b$.

Thus we have

$$a = qb + r, \quad 0 \le r < b.$$

Uniqueness of q and r.

Suppose that a has two representations of the desired form, say

$$a = qb + r = q'b + r' \text{ where}$$

$0 \le r < b$, $0 \le r' < b$

Then $r' - r = qb - q'b = b(q - q')$ — (*)

$$\Rightarrow |r' - r| = |b(q-q')| = |b|\,|q-q'|$$

$$= b\,|q - q'| \quad (\because b > 0 \text{ so } |b| = b)$$ — (I)

Now, $0 \le r < b \Rightarrow -b < -r \le 0$

and $0 \le r' < b$

Adding these two inequalities we get

$$-b < r' - r < b$$

$$\Rightarrow |r' - r| < b$$

(I) $\Rightarrow b|q - q'| = |r' - r| < b$

$$\Rightarrow b|q - q'| < b \Rightarrow |q - q'| < 1$$

$$\Rightarrow 0 \le |q - q'| < 1 \quad \text{— (II)}$$

But since q, q' are integers so $|q - q'|$ is nonnegative integers. So (II) is possible only when $|q - q'| = 0 \Rightarrow q - q' = 0$

$$\Rightarrow q = q'$$

Then from (*)

$$r' - r = b(q - q') = b \cdot 0 \quad [\because q = q']$$

$$\Rightarrow r' - r = 0 \Rightarrow r' = r.$$

Hence both the representations of a are the same. Hence the proof.