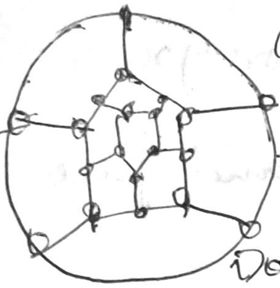
1. F. Harary – ~~Graph~~ ~~har~~ theory (1969)
2. S. A. choudum – A first course in Graph theory.
3. R. J. Wilson, Introduction to G. T.

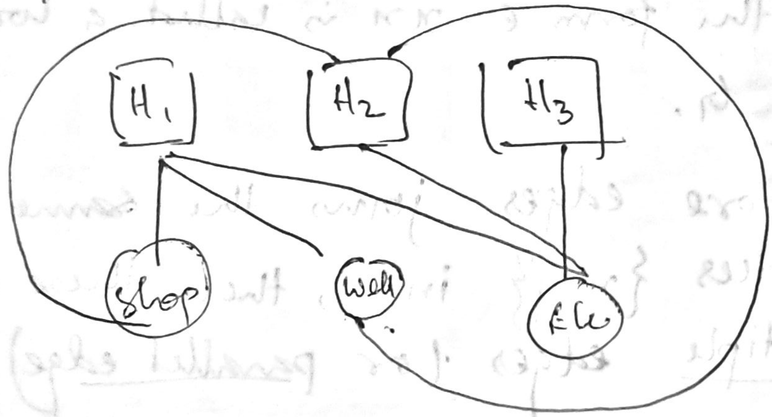
1856

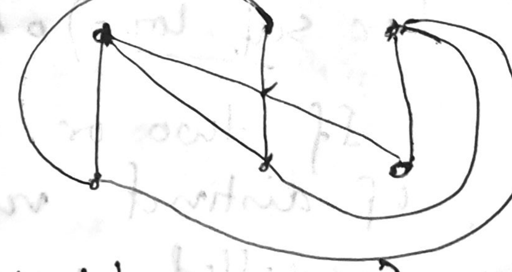


Go around the world

→ hamilton cycle

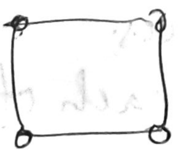
Dodecahedron.





Utility probl.

non planar. graph. (which can drawn without intersecting the edge.



→ planar graph

Ramsey th^m.

A graph G is an ordered pair (V, E) where V is a ~~verte~~ non empty set ~~of~~ objects called vertices and E is a set (possibly empty) of unordered pairs of elts of V.

sometimes this graph G is denoted by G(V, E) or G = (V, E)

$V = V(G)$ is called the vertex set of G
$E = E(G)$ is called edge set of G.

If $e = \{x, y\}$ is an edge of G, that is $e \in E(G)$
then e is also written as $e = xy$ or yx

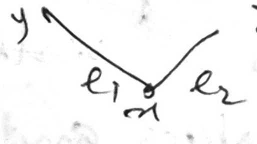
e is said to join x & y and x and y are called the ends of e.

An edge e of the form $e = xx$ is called a loop (or self loop) of G.

If two or more edges joins the same pair of distinct vertices $\{x, y\}$ in G, then these edges are called multiple edges (or parallel edge)

Let $e = xy$ is edge in G
x & y are called adjacent vertices.
e and x are called incident to each other.
e and y " " " " " ,

Let e_1 and e_2 be 2 distinct edges in G.
If e_1 and e_2 are incident with a common vertex x, then we say that e_1 & e_2 are called adjacent edges.



⊛ If a graph G allows only multiple edges but not loops then G is called multigraph.

(*) If a gph, G allows both loops and multiple edges, then G is called pseudograph.

(*) no loops, no multiple ~~graphs~~ edges is called simple graph.

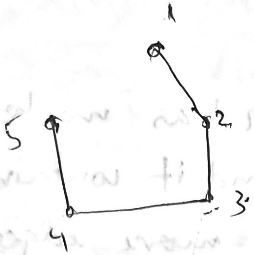
28/07/25

Ex Represent the graph $G(V,E)$

$V = \{1, 2, 3, 4, 5\}$, $E = \{\{x,y\} : |x-y| = 1\}$

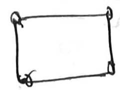
$= \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}\}$

$= \{12, 23, 34, 45\}$



A graph G having p vertices and q edges is called a (P, Q) graph and is denoted by $G(P, Q)$

eg G:



(4, 4) graph

$|V(G)|$ is called order of G. and denoted by p or $p(G)$ or p_G

$|E(G)|$ is called size of graph G and denoted by q or $q(G)$ or q_G

(1, 0) - graph is called trivial graph eg. •

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Ex Show that the size of simple graph of order p cannot exceed $\frac{p(p-1)}{2}$.

Sol^n: Let G be a simple graph of order p

then $|V(G)| = p$.

The elts of $E(G)$ are distinct pairs of elts of $V(G)$

the no. of ways to chose 2 vertices from p vertices is $p_{C_2} = \frac{p(p-1)}{2}$.

Since G is simple, G has no loops and no multi edges and it contains 0 edges or 1 edge or more edges. But G contains at most $p_{C_2} = \frac{p(p-1)}{2}$,

$\therefore 0 \leq |E(G)| \leq \frac{p(p-1)}{2}$

Def^n

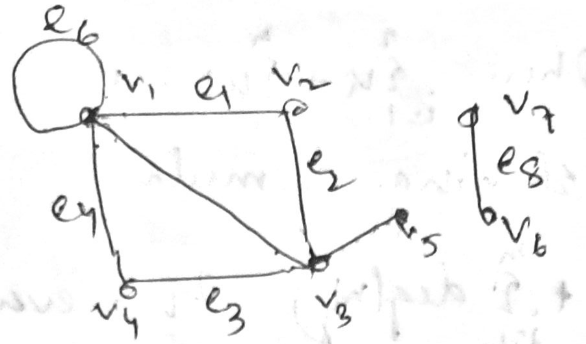
Degree of a vertex v in a graph G is the number of edges of G incident to v

each loop counting as 2 edges in G and is denoted by $\deg(v)$ or $d(v)$

$\delta(G)$ denotes the minimum degree in G

$\Delta(G)$ denotes the maximum " " "

G:



$V(G) = \{v_1, v_2, \dots v_7\}$

$E(G) = \{e_1, e_2, \dots e_8\}$

$d(v_1) = 5 \quad d(v_2) = 2 \quad d(v_3) = 4 \quad d(v_4) = 2$

$d(v_5) = 1 \quad d(v_6) = 1, \quad d(v_7) = 1$

$\delta(G) = 1 \quad \Delta(G) = 5$

$d(v_1) + d(v_2) + d(v_3) + \dots + d(v_7) = 16$

$\Rightarrow \sum_{i=1}^{7} d(v_i) = 2 \times 8 = 2q$

24/07/15

<u>Hand shaking lemma:</u> (1st thm of G.T)

Let G be a (p,q)-graph. Then the $\sum_{i=1}^{p} d(v_i) = 2q$

The sum of degree of vertices of graph G is ~~equal to~~ twice the no. of edges.

<u>Ans:</u> ~~Let G~~ Notice that every edge v_iv_j in G is incident with 2 vertices v_i & v_j. When the degree of vertices of G are summed the edge v_iv_j is counted once at v_i and once at vertex v_j. Thus v_iv_j is counted twice in the sum $\sum_{i=1}^{p} deg(v_i) = \sum_{i=1}^{n} deg(v_i) = 2q$.

<u>Cor:</u> The no. of vertices of odd degree is even in any (P,q) graph

<u>Ans:</u> Let us a graph (P,q) has n odd degree vertices $v_1, v_2 \dots v_n$, and m even degree vertices.

$u_1, u_2, \dots, u_m$, then $\sum_{i=1}^{n} v_i + \sum_{i=1}^{m} u_i = p$.

Now, by handshaking formula

$$\sum_{i=1}^{n} deg(v_i) + \sum_{j=1}^{m} deg(u_j) = 2q = \text{even integer.} \longrightarrow (1)$$

Since each deg of u_i is even, so it follows that $\sum_{j=1}^{m} deg(u_j)$ is an even integer

Now the sum of odd no. of odd integer is odd integer.

So from (1) we have $\sum_{i=1}^{n} d(v_i)$ is even which is ~~true~~ only if n is even.

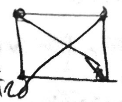
Hence in a graph there are even no. of ~~de~~ vertices of odd degree.

A complete graph of order n:

(1) each pair of distinct vertex

A simple graph in which each pair of distinct vertex is joined by an edge is called a complete graph.

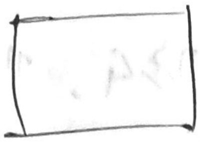
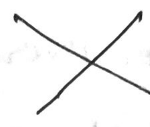
K_4:



A complete graph of order n is denoted by K_n.

$$|E(K_p)| = \frac{p(p-1)}{2}.$$

The complement $\bar{G}$ of a simple graph G is the simple graph with $V(\bar{G}) = V(G)$ and any two ~~vertices~~ vertices in $\bar{G}$ are adjacent iff they are not adjacent in G.

G :  $\overline{G}$: 

H :  $\overline{H}$ = 

If G is simple graph of order p and size q Then $|V(G)| = |V(\overline{G})| = q$
$|E(G)| + |E(\overline{G})| = \frac{p(p-1)}{2}$

Regular gr: A group G is said to be regular if each vertex of G has same degree.

r-regular: A gr. G is called r-regular if every vertex of G has degree r.

Ex! If $G(P,q)$ be a r-regular graph then find q.

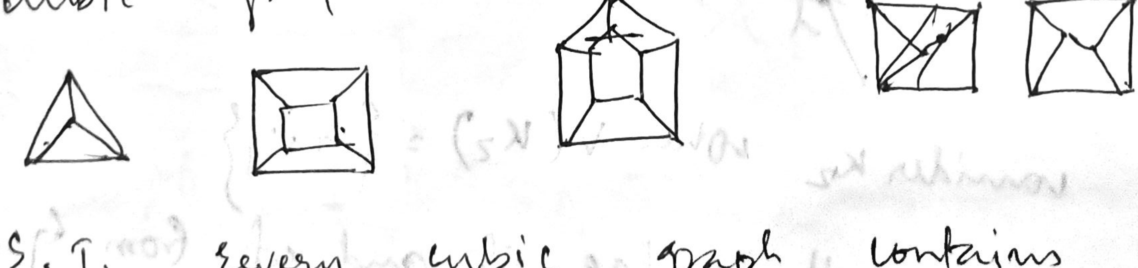
Sol^n: Since $G(P,q)$ is a r-regular graph

$\therefore \deg(v_i) = r, \quad \forall v_i \in V(G)$

$$\therefore \sum_{i=1}^{p} \deg(v_i) = 2q.$$

$$\Rightarrow rp = 2q \Rightarrow q = \frac{rp}{2}$$

Cubic Graph: A 3-regular graph is called a cubic graph.



Ex S.T. every cubic graph contains even no. of vertices

Sol^n: Apply cor-I.

Ex Does $\exists$ a 4-regular graph of order 6.

8

Ex. S.T. for every even int $n \geq 4$, there exist a 3-regular simple graph of order n.

Ans: we prove this result by induction on n.

if $n=4$ then $K_4 =$ 

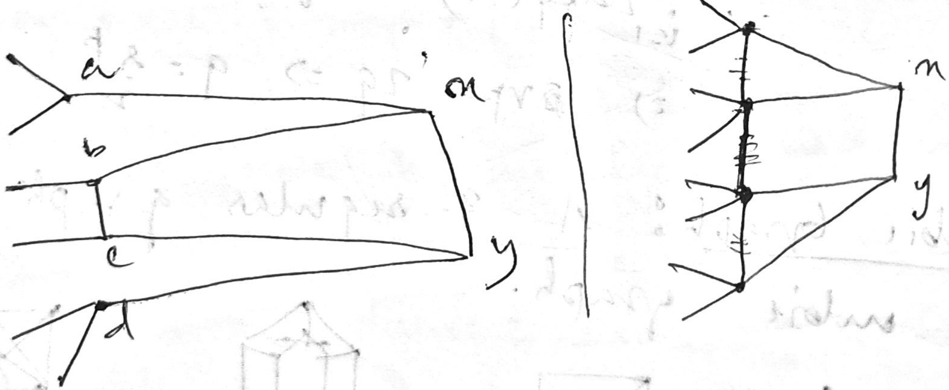
is the only graph which is 3-regular of order 4.

Assume that the result is true for all even $m<n$

~~conversely~~

consider a 3-regular simple graph G of order $m = n-2$.

Since $n \geq 6$ $\exists$ 4 vertices a, b, c, d ~~s.t.~~ ~~each have degree 3~~. such that each have degree 3 and $ab, bc, cd \in E(G)$



consider K_2 when $V(K_2) = \{x, y\}$

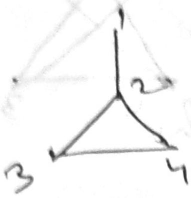
Remove the edge ab and cd from G and add the edges ax, bx, cy and dy

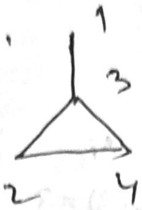
This results a graph G' which is clearly a 3-regular graph of order n

$G = (V(G), E(G))$

$H = (V(H), E(H))$.

identical
$V(G) = V(H)$
$E(G) = E(H)$

G:


H:


not identical but isomorphic,

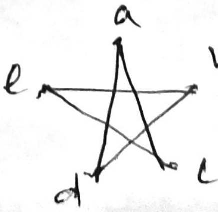
Def^n Two graphs G & H are said to be isomorphic, written $G \cong H$ or $G = H$ only for graph

if $\exists$ are two bijective maps $\theta : V(G) \longrightarrow V(H)$ and $\phi : E(G) \longrightarrow E(H)$ s.t. $e = uv \in E(G)$

$\Leftrightarrow \phi(e) = \theta(u)\theta(v) \in E(H)$ ~~then~~

A pair (θ, ϕ) ~~is called~~ of mapping is called an isomorphism betn G & H

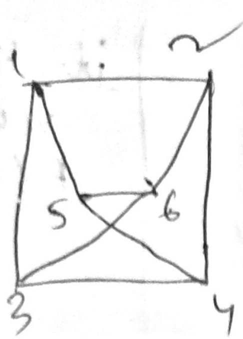
Ex G:


H:


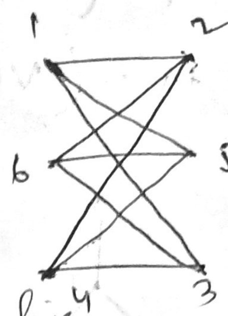
$\therefore \theta(u) = a,$ $\theta(x) = b,$
$\theta(v) = c.$ $\theta(w) = e$
$\theta(y) = d.$

$\phi(uv) = \theta(u)\theta(v) = ac$

G_1



H_1 :

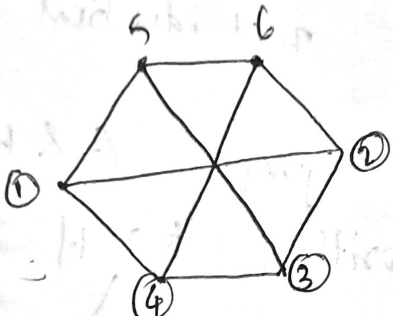


Yes isomorphic

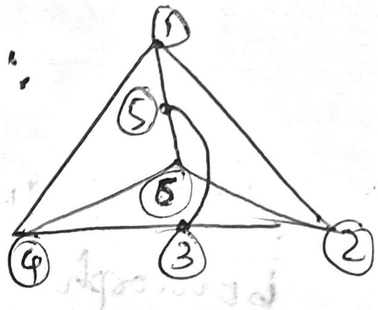
$\theta(1)=1$ $\theta(2)=2$ $\theta(3)=3$
$\theta(5)=5$, $\theta(6)=6$ $\theta(4)=4$

17/08/15

① G' :



H' :

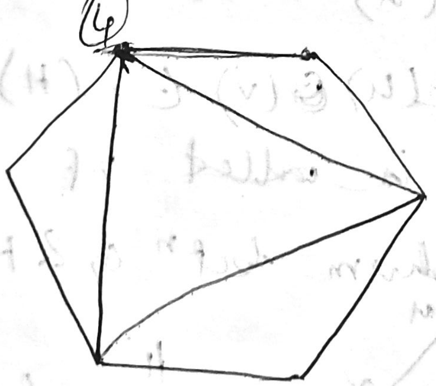


$\phi(15) = \theta(1)\theta(5)$
$= 14$

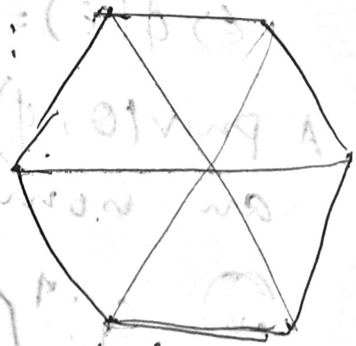
$G' \cong H$. $\theta(1)=1$
$\theta(5)=5$
$\theta(2)=2$
$\theta(4)=4$

②

G_1 :



H_1 :



$G_1 \ncong H_1$ all degree are ③

③ G_2 :

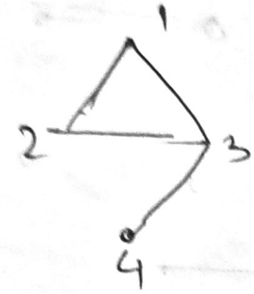
H_2 :

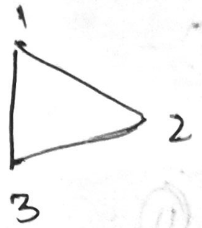
$G_2 \ncong H_2$

Subgph :

A gph H is called a subgp of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and is denoted by $H \subseteq G$

eg.

G:


H_1 :


H_2 :

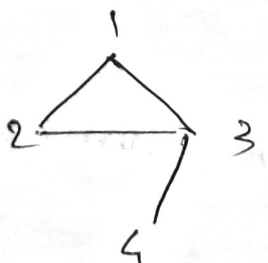

② if $H \subseteq G$ and $H \neq G$, then H is called a proper subgp.

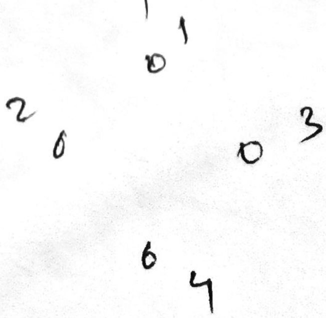
trivial is also proper subgp.

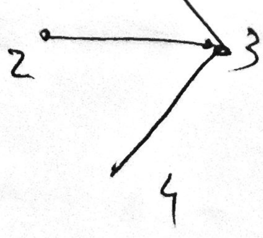
graph itself is not a proper subgp.

Types of subgp :

1. Let G be a gp. A graph H is called a spanning subgp. if $V(H) = V(G)$ and $E(H) \subseteq E(G)$

G:


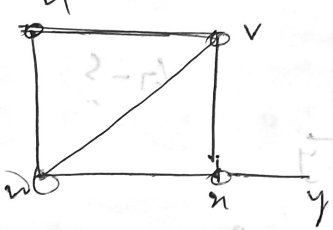
H_1 :


H_2 :


(14) $S \subseteq V(G)$; $S \neq \phi$ at least one vertex

The induced subgph. of G generated by a given set S is denoted by $\langle S \rangle$ is the subgph. of G s.t. $V(\langle S \rangle) = S$ and any 2 vertices u & v of $\langle S \rangle$ are adjacent iff u and v are adjacent in G. Thus $\langle S \rangle$ is maximal subgph of G having vertex set S

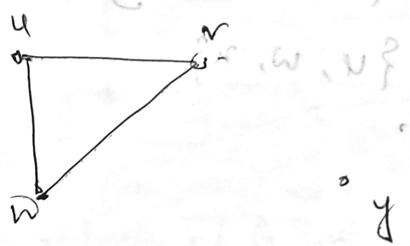
(1) G:



$V(G) = \{u, v, w, x, y\}$

$S = \{u, v, w, y\}$

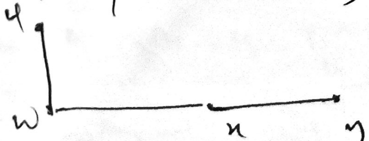
$\langle S \rangle =$



(2)

G: $S_1 = \{u, w, x, y\} \subset V(G)$

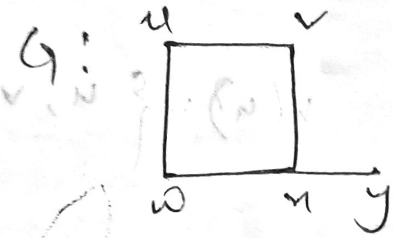
$\langle S_1 \rangle$:

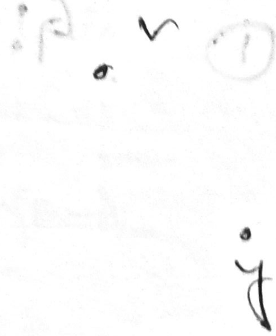


(2) The induced subgp. of G generated by V(G) − S is denoted by G − S is subgp of G obtained from G by deleting the vertices in S together with their incident edges.

if S = {u} then G − S = G − u

Ex





V(G) = {u, v, w, x, y}

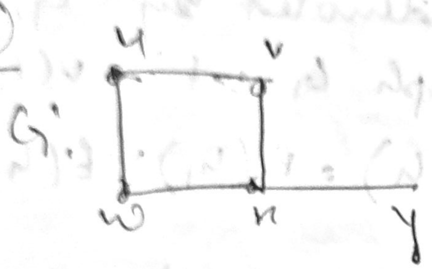
S = {u, w, x}.

Let $K \neq \phi$; $K \subseteq E(G)$

The edge induced subgp of G generated by K, is denoted by $\langle K \rangle$ is the subgraph G s.t. $V(\langle K \rangle)$ = set of all ends in K.

$= \{v : v \text{ is an end of } e \in K\}$.

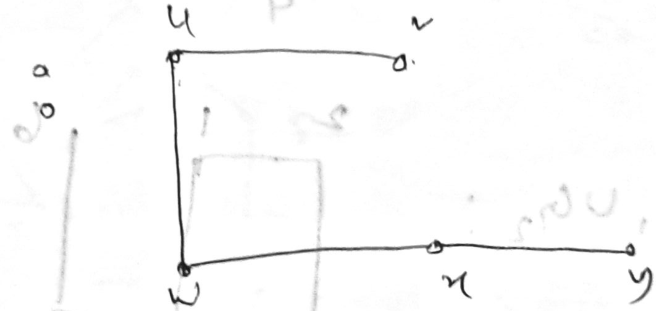
Ex



$E(G) = \{uv, vx, uw, vw, wx, xy\}$

$K = \{uv, uw, xy, wx\}$.

$(\langle K \rangle)$:

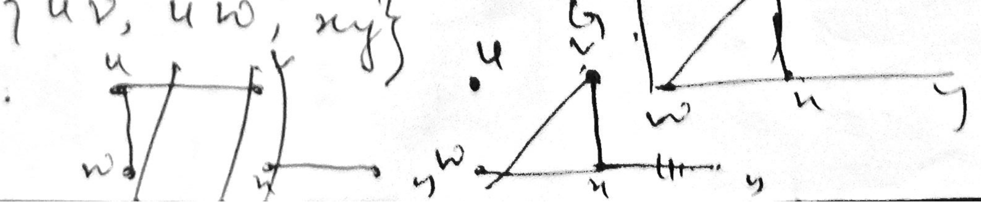


The spanning subgph of G with the edge set $E(G) - K$ is written as $G - K$ is the subgph of G obtaining from G by deleting the edges in K.

If $K = \{e\}$ then $G - K$ is same as $G - e$

$K = \{uv, uw, xy\}$

$G - K$:



Let G_1 and G_2 be any 2 graphs

1. G_1 & G_2 are said to be disjoint if

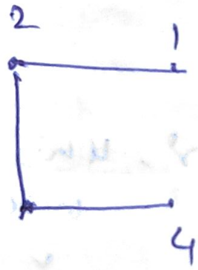
iff $V(G_1) \cap V(G_2) = \phi$

2. G_1 & G_2 are said to edge disjoint if

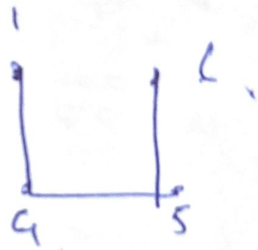
$E(G_1) \cap E(G_2) = \phi$

3. union of G_1 & G_2 denoted by $G_1 \cup G_2$, is defined as the graph G s.t. $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$

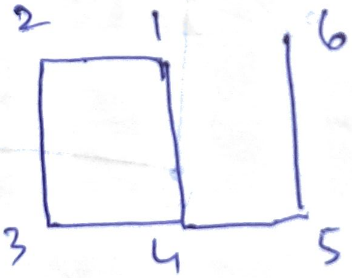
① G_1:



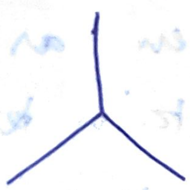
G_2:



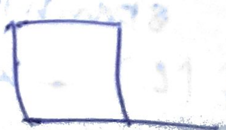
$G_1 \cup G_2$:



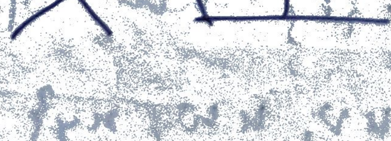
② G:



H:



$G \cup H$



Intersection

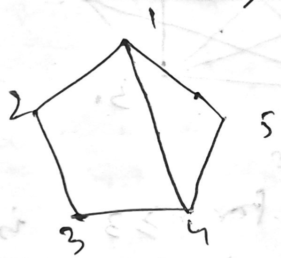
Let $V(G_1) \cap V(G_2) \neq \phi$.

The intersection of G_1 & G_2 denoted by $G_1 \cap G_2$ is defined as the graph G s.t.

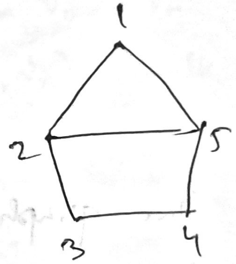
$V(G) = V(G_1) \cap V(G_2)$

& $E(G) = E(G_1) \cap E(G_2)$.

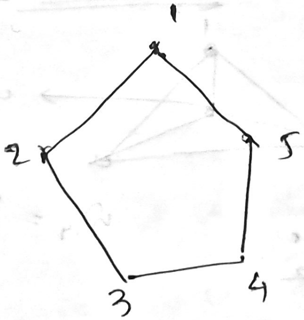
G_1 :



G_2 :



$G_1 \cap G_2$:

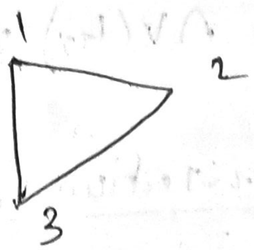


The join (or sum) of two disjoint graphs G_1 & G_2 denoted by $G_1 + G_2$ is the graph G' s.t. $V(G) = V(G_1) \cup V(G_2)$

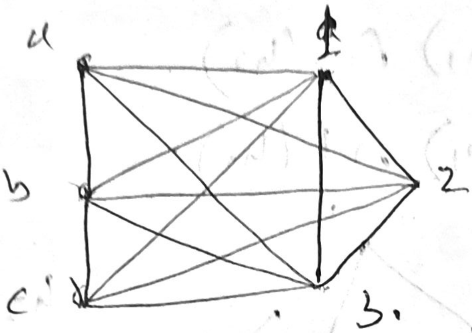
$E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$

G_1 :

G_2 :

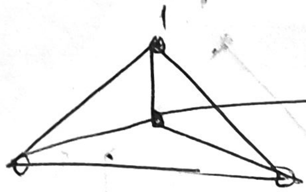


$G_1 + G_2$:

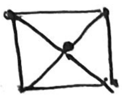


The graph $C_n \overset{\text{sum}}{+} K_1$ for $n \geq 3$ is denoted by W_{1n} is called a wheel on $n+1$ vertices.

W_{13}



→ complete graph of 4 vertices.

$W_{14} =$ 

Ring - sum:

The ring - sum of any 2 graphs G_1 & G_2 denoted by $G_1 \oplus G_2$ is defined as the graph G s.t. $V(G) = V(G_1) \cup V(G_2)$

$$E(G) = \left[E(G_1) \cup E(G_2)\right] - \left[E(G_1) \cap E(G_2)\right]$$

$$= \left[E(G_1) - E(G_2)\right] \cup \left[E(G_2) - E(G_1)\right]$$

$$= E(G_1) \Delta E(G_2)$$

→ symmetric relation.