Corollary:- If a and b are given integers, not both zero, then the set $T = \{ax + by \mid x, y \text{ are integers}\}$ is precisely the set of all multiples of $d = \gcd(a, b)$.

Pf: $d = \gcd(a, b)$

$\Rightarrow d \mid a,\ d \mid b \Rightarrow d \mid (ax + by)$ for all integers x, y.

$\Rightarrow ax + by = ds$, s is an integer.

Thus every number of T is a multiple of d.

Conversely let $d = ax_0 + by_0$ for suitable x_0 and y_0. Then any multiple nd of d is of the form

$$nd = n(ax_0 + by_0) = a(nx_0) + b(ny_0)$$

Hence nd is a linear combination of a and b and by definition it lies in T.

Definition. Two integers a and b, not both of which are zero, are said to be relatively prime whenever $\gcd(a, b) = 1$.

eg. 5 and 8 are relatively prime as $\gcd(5, 8) = 1$

Theorem:- Let a and b be integers, not both zero. Then a and b are relatively prime if and only if $\exists$ integers x and y s.t. $1 = ax + by$.

Pf: Let $d = \gcd(a, b)$.

Since a and b are relatively prime

$\therefore d = \gcd(a, b) = 1$.

As $d = \gcd(a, b)$, so $\exists$ integers x and y s.t.

$$d = ax + by \Rightarrow 1 = ax + by.$$

Conversely let $1 = ax + by$ for some choice of x and y.

As $d = \gcd(a, b)$, so $d \mid a,\ d \mid b$

$\Rightarrow d \mid ax + by$ for every choice of x and y.

$\Rightarrow d \mid 1$.

Since d is positive integer so d must be 1

$\therefore d = \gcd(a, b) = 1 \Rightarrow a$ and b are relatively prime.

Corollary: If $\gcd(a,b)=d$. Then $\gcd\left(\frac{a}{d},\frac{b}{d}\right)=1$.

Pf: Let $d=\gcd(a,b)$. So $d\mid a$. $d\mid b$

$\Rightarrow \frac{a}{d}$ and $\frac{b}{d}$ are integers.

As $d=\gcd(a,b)$ so. $\exists$ integers x and y s.t. $d=ax+by$

$\Rightarrow 1=\left(\frac{a}{d}\right)x+\left(\frac{b}{d}\right)y \Rightarrow \frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

$\Rightarrow \gcd\left(\frac{a}{d},\frac{b}{d}\right)=1$ #

Corollary If $a\mid c$ and $b\mid c$ with $\gcd(a,b)=1$, $ab\mid c$.

Pf: Let $a\mid c$ and $b\mid c$ and $\gcd(a,b)=1$.

$\therefore c=ar$ and $c=bs$ for some integers r and s.

Also $\gcd(a,b)=1$.

So $\exists$ integers x and y s.t. $1=ax+by$.

Now $c=c\cdot 1=c\cdot(ax+by)=acx+bcy$

$\Rightarrow c=abrx+bsxy=ab(rx+sy)$

$\Rightarrow ab\mid c$ #

Theorem: If $a\mid bc$, with $\gcd(a,b)=1$ then $a\mid c$

Pf Since $\gcd(a,b)=1$, so $\exists$ integers x and y such that $1=ax+by$

Also, $c=1\cdot c=(ax+by)c=acx+bcy$

But $a\mid bc$ and $a\mid ac$ so $a\mid acx+bcy$

$\Rightarrow a\mid c$

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