(5) $\Delta^2 \equiv E^2 - 2E + 1$

Proof: $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$$= \Delta\{f(x+h) - f(x)\}$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$= E^2 f(x) - 2E f(x) + f(x)$$

$$= (E^2 - 2E + 1) f(x)$$

$\therefore \Delta^2 \equiv E^2 - 2E + 1$

(6) $(1+\Delta)(1-\nabla) \equiv 1$

Proof: $(1+\Delta)(1-\nabla) f(x) = (1+\Delta)\{(1-\nabla) f(x)\}$

$$= (1+\Delta)\{f(x) - \nabla f(x)\}$$

$$= (1+\Delta)\left[f(x) - \{f(x) - f(x-h)\}\right] \cancel{- f(x)}$$

$$= (1+\Delta)[\cancel{f(x)}\ f(x-h)]$$

$$= E f(x-h) \qquad [\because 1+\Delta \equiv E]$$

$$= f(x)$$

$\Rightarrow (1+\Delta)(1-\nabla) = 1 . f(x)$

$\therefore (1+\Delta)(1-\nabla) \equiv 1$ //

(7) $(\Delta - \nabla) \equiv \Delta\nabla$

$\cancel{\nabla\Delta} \equiv \nabla\Delta$

$\nabla\Delta \equiv \Delta\nabla$

Proof: We have

$$\Delta\nabla f(x) = \Delta[f(x) - f(x-h)]$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= \{f(x+h) - f(x)\} - \{f(x) - f(x-h)\}$$

$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$$

$\therefore \Delta\nabla \equiv \Delta - \nabla$ //

Factorial Notation:

The continued product of m factors:

$x(x-h)(x-2h) \cdots \{x-(m-1)h\}$ in which the first factor is x and the successive factors decrease by a constant difference (say h) is called a factorial function and is denoted by $x^{(m)}$, where m is a positive integer.

Thus
$$x^{(1)} = x \; ; \quad x^{(2)} = x(x-h) \; ; \quad x^{(3)} = x(x-h)(x-2h)$$
and so on.

In general
$$x^{(m)} = x(x-h)(x-2h) \cdots \{x-(m-1)h\}$$

As a particular case when $h=1$, we define

$$x^{(1)} = x \; ; \quad x^{(2)} = x(x-1) \; ; \quad x^{(3)} = x(x-1)(x-2) \; ; \cdots$$

and
$$x^{(m)} = x(x-1)(x-2) \cdots \{x-(m-1)\}$$
$$= \frac{x(x-1)(x-2) \cdots \{x-(m-1)\}\,(x-m)!}{(x-m)!}$$
$$= \frac{x!}{(x-m)!}, \quad x > m$$

The function $f(x) = x^{(m)} = x(x-h)(x-2h) \cdots \{x-(m-1)h\}, m>0$ is called the factorial polynomial of degree m.

* Differences of a factorial Polynomial:

$$\Delta^m x^{(m)} = m!\, h^m \quad \text{and} \quad \Delta^{m+1} x^{(m)} = 0$$

Evaluate ;

(1) $\Delta\,(x^2+e^x+2)$

(2) $\dfrac{\Delta^2}{E}x^3$

(3) $\dfrac{\Delta^2 x^3}{Ex^3}$

(4) $\Delta^3(ax^3+bx^2+cx+d)$

(5) $\Delta \tan^{-1} ax$

(1) Solⁿ: We have,

$$\Delta f(x) = f(x+h) - f(x)$$

$$\therefore\ \Delta x^2 = (x+h)^2 - x^2 = h^2 + 2hx$$

$$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$$

$$\Delta 2 = 2\Delta(1) = 2.(1-1) = 0$$

$$\therefore\ \Delta\,(x^2+e^x+2) = \Delta x^2 + \Delta e^x + \Delta 2$$

$$= h^2 + 2hx + e^x(e^h - 1) + 0$$

(2) Solⁿ

$$\frac{\Delta^2}{E}x^3 = \frac{(E-1)^2}{E}x^3$$

$$= \left[\frac{E^2+1-2E}{E}\right]x^3$$

$$= \left[E + E^{-1} - 2\right]x^3$$

$$= Ex^3 + E^{-1}x^3 - 2x^3 \quad \text{[taking } h=1\text{]}$$

$$= (x+1)^3 + (x-1)^3 - 2x^3$$

$$= 6x$$

(3) Solⁿ:

$$\frac{\Delta^2 x^3}{Ex^3} = \frac{(E-1)^2x^3}{Ex^3} = \frac{(E^2+1-2E)x^3}{Ex^3}$$

$$= \frac{E^2x^3 + x^3 - 2Ex^3}{Ex^3} = \frac{(x+2)^3 + (x^3) - 2(x+1)^3}{(x+1)^3}$$

$$= \frac{6}{(x+1)^2}$$

(4) Solⁿ:-

$$\Delta^3(ax^3+bx^2+cx+d)$$
$$= \Delta^3 ax^3$$
$$= a\,\Delta^3 x^3$$
$$= a \cdot 3!\, h^3$$
$$= 6ah^3 \quad //$$

(5) Solⁿ: $\Delta \tan^{-1} ax$

$$= \tan^{-1} a(x+1) - \tan^{-1} ax$$
$$= \tan^{-1}\left[\frac{ax+a-ax}{1+a(x+1)ax}\right]$$
$$= \tan^{-1}\left[\frac{a}{1+a^2x^2+a^2x}\right] \quad //$$

Ex) Evaluate

(i) $\Delta^3 (1-x)(1-2x)(1-3x)$

(ii) $\Delta^n (e^{ax+b})$, the interval of differencing being unity.

(i) Solⁿ: Here, $f(x) = (1-x)(1-2x)(1-3x)$
$$= -6x^3+11x^2-6x+1$$

$$\therefore \Delta^3(-6x^3+11x^2-6x+1)$$
$$= -6\Delta^3 x^3 + 11\Delta^3 x^2 - 6\Delta^3 x + \Delta^3 1$$
$$= -6 \cdot 3!$$
$$= -36 \quad //$$

(ii) Solⁿ: $\Delta(e^{ax+b}) =$

$$= e^{a(x+1)+b} - e^{a(x)+b}$$
$$= e^{ax+b}(e^a-1)$$

$$\Delta^2 e^{(ax+b)} = \Delta(\Delta e^{ax+b})$$
$$= \Delta\{e^{ax+b}(e^a-1)\}$$
$$= (e^a-1)\,\Delta e^{(ax+b)} = (e^a-1)^2 e^{ax+b}$$

Proceeding in a similar way, we get,

$$\Delta^n \left(e^{ax+b}\right) = e^{ax+b} \left(e^a - 1\right)^n$$

Ex) Prove that $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{E e^x}{\Delta^2 e^x}$, the interval of differencing being h.

Soln: We have,

$$E e^x = e^{x+h}$$

$$\Delta e^x = e^{x+h} - e^x = e^x (e^h - 1)$$

$$\Delta^2 e^x = e^x (e^h - 1)^2$$

$\therefore$ $\left(\frac{\Delta^2}{E}\right) e^x = (\Delta^2 E^{-1}) e^x$

$$= \Delta^2 e^{x-h}$$

$$= e^{-h} \Delta^2 e^x$$

$$= e^{-h} \cdot e^x (e^h - 1)^2$$

$\therefore$ R.H.S. $= \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{E e^x}{\Delta^2 e^x}$

$$= e^{x-h} (e^h - 1)^2 \cdot \frac{e^{x+h}}{e^x (e^h - 1)^2}$$

$$= e^x = \text{L.H.S.}$$

//

Ex) Evaluate $\Delta^2 (\cos 2x)$: Ans: $-4 \sin^2 h \cos(2x + 2h)$

Ex) Show that $\Delta \log f(x) = \log \left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$

Soln: We have,

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log \left[\frac{E f(x)}{f(x)}\right]$$

$$= \log \left[\frac{(1+\Delta) f(x)}{f(x)}\right]$$

$$= \log \left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log \left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

//

Ex) Find $\Delta\left(\frac{x^2}{\cos 2x}\right)$

Sol^n:- take $f(x) = x^2$

$g(x) = \cos 2x$

$\therefore \quad \Delta f(x) = (x+h)^2 - x^2 = h(2x+h)$

$\Delta g(x) = \cos 2(x+h) - \cos 2x$

$= -2\sin(2x+h)\sin h$

Now, $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)\, g(x)}$

$= \frac{\cos 2x \cdot h(2x+h) - x^2\{-2\sin(2x+h)\sin h\}}{\cos(2x+2h)\cos 2x}$

$$= \frac{h(2x+h)\cos 2x + 2x^2 \sin h \sin(2x+h)}{\cos(2x+2h)\cos 2x} \quad //$$

Ex) Evaluate: $\Delta\left(\frac{2^x}{(x+1)!}\right)$; $h=1$

<u>Ans:-</u> $-\frac{x}{(x+2)!}\,2^x$

Ex) Evaluate $\left(\frac{\Delta^2}{E}\right)x^3$: Ans: $6x$

[take $h = 1$]