Supremum or least upper bound

If S is bounded above, then a number u is said to be supremum (or least upper bound) of S if it satisfies the conditions:

1. u is an upper bound of S, and
2. if v is any upper bound of S then $u \leq v$.

eg. $S = \{1, 2, 3, 4\}$

Then supremum of $S = 4$.

eg. $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Then Supremum of $S = 1$.

Infimum or greatest lower bound

If S is bounded below, then a number w is said to be an infimum (or greatest lower bound) of S if it satisfies the condition:

1. w is a lower bound of S, and
2. if t is any lower bound of S, then $t \leq w$.

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eg. $S = \{1, 2, 3, 4\}$
infimum of $S = 1$.

<u>Note</u>: ~~If~~ A subset S of $\mathbb{R}$ can have atmost one supremum.

<u>Pf</u>: If S has no supremum then there is nothing to prove. Suppose supremum of S exists. If possible let there exist two supremum of S, say u_1 and u_2.

Now, if $u_1 < u_2$ then as u_2 is a supremum (or least upper bound) of S so u_1 can not be an upper bound of S, which is a contra-diction to our assumption as u_1 is a supremum of S. So u_1 can not be less than u_2. Similarly ~~if~~ $u_2 < u_1$ is also impossible. So we must have $u_1 = u_2$. Hence there can not be more than supremum of a subset S of $\mathbb{R}$.

<u>Note</u>. A subset S of $\mathbb{R}$ can have atmost one infimum.

<u>Note</u>: If infimum and supremum of a set S exist, we will denote them by $\sup S$ and $\inf S$.

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Ex. Let $S_1 = \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.

Sol^n: 1st part

$S_1 = \{x \in \mathbb{R} : x \geq 0\}$

clearly 0 is less than or equal to every element of S_1. So 0 is a lower bound of S_1.

2nd part

If possible let u be an upper bound of S_1. Then $x \leq u$ for all $x \in S_1$

$\Rightarrow u \geq 0$. as $x \geq 0$

$\therefore u+1 > 0 \Rightarrow u+1 \in S_1$.

But $u < u+1$ ~~as~~.

~~$\Rightarrow u$ can not be an upper bound of~~

As $u+1 \in S_1$ so u can not be an upper bound of S_1. Hence S_1 has no upper bounds.

3rd part: We have seen that 0 is a lower bound of S_1. Let ω be any arbitrary lower bound of S_1. Then $\omega \leq x$ for all $x \in S_1$.

Since $0 \in S_1$ so, $\omega \leq 0$.

$\therefore$ ~~the lower~~

$\therefore$ 0 is the greatest lower bound of S_1

$\Rightarrow \inf S_1 = 0$.

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