## Minors and Co-factors:

**Minors:** If we delete the ith row and the jth column passing through the element $a_{ij}$ of the matrix A of order n, then the determinant of the square sub-matrix of order (n-1) obtained is called the minor of the element $a_{ij}$ and denoted by $M_{ij}$.

Let us consider a matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

**Co-factor of an element of a Matrix:**

If the minor $M_{ij}$ multiplied by $(-1)^{i+j}$, then it is called the Co-factor of the element $a_{ij}$. It is denoted by $A_{ij}$ or $C_{ij}$.

The Co-factor of the element

$$a_{21} = A_{21} = (-1)^{1+2} M_{21}$$

$$= - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

## Adjoint of a square Matrix:

Let $A = [a_{ij}]$ be any $n \times n$ matrix, then adjoint of $A$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where $A_{ij}$ denotes the co-factor of the element $a_{ij}$ in the determinant $|A|$. The adjoint of the matrix $A$ is denoted by the symbol adj(A).

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a $3 \times 3$ matrix,

then co-factor matrix or $cof(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\therefore adj(A) = [cof(A)]' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

## Inverse of a Matrix:

A square matrix $A$ of order $n$ is said to be invertible, if $\exists$ a square matrix $B$ of order $n$, such that

$$AB = BA = I$$

$B$ is called the inverse of $A$, we have $B = A^{-1}$.

The inverse of $A$ is given by

$$A^{-1} = \frac{adj(A)}{|A|}, \text{ Provided } |A| \neq 0$$

i.e., the necessary and sufficient condition for a square matrix $A$ to possess the inverse is that $|A| \neq 0$.

5) * Working rule to find the inverse of a Matrix by elementary Row Transformation:

(1) Write the given matrix as

$$A = I_n A \quad \text{-----} (1)$$

where $I_n$ is the identity matrix of order n, which is equal to the order of the matrix A.

(ii) Perform elementary row transformation on A of left hand side of eqn (1) to reduce it into the identity matrix.

(iii) Apply the same elementary row transformation on the I of the right hand side, so that I is reduced to B.

(iv) The transformation reduces the eqn (1) i.e.

$$A = I_n A \quad \text{into} \quad I_n = BA,$$

where B denotes the inverse of A.

Ex) Using elementary transformation, find the inverse of

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(a) Soln: Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Now, $A = I_2 A$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \qquad R_2 \to R_2 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad R_2 \to \frac{1}{5} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/5 & 1/5 \end{bmatrix} A \qquad R_1 \to R_1 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}$$

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(b) Sol$^n$:- Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

Now, $A = I_2 A$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \qquad R_2 \to R_2 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \qquad R_1 \to R_1 - 3R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

(d) Sol$^n$:- Let $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Now, $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_2 \to R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + 2R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \qquad R_1 \to R_1 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \quad //$$