Theorem: Finite integral domains are fields.

Pf. Let D be a finite integral domain and following are all of its elements,

$$0, 1, a_1, a_2, \ldots, a_n$$

To show D is a field it is sufficient to show that every nonzero element of D have multiplicative inverse.

Let $a (\neq 0) \in D$. Now Consider the following sequence of elements

$$a1, aa_1, aa_2, \ldots, aa_n$$ elements of D

All these elements are distinct, because

$$\text{for } a a_i = a a_j$$

$$\Rightarrow a_i = a_j \text{ [by cancellation law]}$$

Again all these elements $a1, aa_1, \ldots, aa_n$ are nonzero as they are elements of an integral domain (integral domain has no zero divisor i.e. product of two non zero element cannot be zero).

Since D is finite so the elements $a1, aa_1, aa_2, \ldots aa_n$ are the same elements as $1, a_1, a_2, \ldots, a_n$ in some order.

So we can conclude that

either $a1 = 1$

$\Rightarrow a1 = 1.1 \Rightarrow a = 1$

or

$aa_j = 1$ for some i, $1 \le i \le n$

Thus if $a = 1$, a is its own inverse.

otherwise $\exists$ some $a_j \in D$ s.t. $aa_j = 1 = a_j a$.

$\Rightarrow a_j$ is the inverse of a,

$\therefore$ every non zero element a of D has multiplicative inverse. Hence D is a field.