Ex) Solve:

$$(x-2y+1)\,dx + (4x-3y-6)\,dy = 0 \quad \cdots (1)$$

Sol^n:

Where $a_1 = 1,\ b_1 = -2,\ c_1 = 1$

and $a_2 = 4,\ b_2 = -3,\ c_2 = -6$ and so

$$\frac{a_2}{a_1} = 4 \quad \text{and} \quad \frac{b_2}{b_1} = \frac{3}{2}$$

$$\therefore \quad \frac{a_2}{a_1} \neq \frac{b_2}{b_1}$$

Now, we make the transformation

$$x = X + h$$
$$y = Y + k$$

where (h,k) is the solution of the system

$$h - 2k + 1 = 0$$
$$4h - 3k - 6 = 0$$

The solution of this system is $h = 3,\ k = 2$ and so the transformation is

$$x = X + 3$$
$$y = Y + 2$$

This reduces eq^n(1) to the homogeneous equation

$$(X+3-2Y-4+1)\,dX + (4X+12-3Y-6-6)\,dY = 0$$

$$\Rightarrow (X-2Y)\,dX + (4X-3Y)\,dY = 0 \quad \cdots (2)$$

We first put this homogeneous eq^n in the form

$$\frac{dY}{dX} = \frac{1-2(Y/X)}{3(Y/X)-4}$$

and let $Y = vX$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\therefore \quad v + X\frac{dv}{dX} = \frac{1-2v}{3v-4}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-2v}{3v-4} - v = \frac{-3v^2+2v+1}{3v-4}$$

$$\Rightarrow \frac{(3v-4)\,dv}{3v^2-2v-1} = -\frac{dX}{X} \quad \cdots (4)$$

Integrating, we obtain

$$\frac{1}{2}\ln|3v^2-2v-1| - \frac{3}{4}\ln\left|\frac{3v-3}{3v+1}\right| = -\ln x + \ln c_1$$

$$\Rightarrow 2\ln|3v^2-2v-1| - 3\ln\left|\frac{3v-3}{3v+1}\right| = -4\ln x + 4\ln c_1$$

$$\Rightarrow \ln(3v^2-2v-1)^2 - \ln\left(\frac{3v-3}{3v+1}\right)^3 = \ln\left(\frac{c_1^4}{x^4}\right)$$

$$\Rightarrow \ln\left|\frac{(3v+1)^5}{v-1}\right| = \ln\left(\frac{c_1^4}{x^4}\right)$$

$$\Rightarrow x^4\left|(3v+1)^5\right| = c\left|v-1\right|$$

where $c = c_1^4$

$$\Rightarrow x^4\left|\left(3\frac{y}{x}+1\right)^5\right| = c\left|\frac{y}{x}-1\right|$$

$$\Rightarrow |3y+x|^5 = c|y-x|$$

$$\Rightarrow |3(y-2)+x-3|^5 = c|y-2-x+3|$$

$$\Rightarrow |x+3y-9|^5 = c|y-x+1|$$

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