Thm:- The nth divided differences of a polynomial of the nth degree are constant.

Newton's Divided difference Formula:

Consider the f^n $y=f(x)$, let $f(x_0), f(x_1), ---, f(x_n)$ be the values of the f^n $f(x)$ corresponding to the arguments $x_0, x_1, x_2, ---, x_n$ given at unequal intervals:

Then, we have,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow (x - x_0) f(x, x_0) = f(x) - f(x_0)$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad ---- (1)$$

and $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad --- (2)$$

Similarly, we have

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \quad --- (3)$$

$$\vdots$$

$$f(x, x_0, x_1, ---, x_{n-1}) = f(x_0, x_1, ---, x_n) + (x - x_n) f(x, x_0, x_1, ---, x_n) \quad --- (4)$$

Multiplying eqⁿ (2) by $(x-x_0)$, eqⁿ (3) by $(x-x_0)(x-x_1)$ ----- and eqⁿ (4) by $(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ and adding to eqⁿ (1), we get,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\ldots,x_n)$$
$$+ R_n$$

where the remainder R_n is given by

$$R_n = (x-x_0)(x-x_1)\cdots(x-x_n)f(x,x_0,x_1,\ldots,x_n)$$

In case we assume the given fⁿ $f(x)$ expressible as a polynomial of nth degree, then $f(x,x_0,x_1,\ldots,x_n)$ vanishes so that $R_n = 0$.

Then we have,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\ldots,x_n)$$

This formula is called Newton's divided difference interpolation formula for unequal intervals.

Ex) By means of Newton's divided difference formula find the value of $f(8)$ and $f(15)$ from the following table:

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

$\underline{Sol^n}$:- The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$	
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Now from Newton's divided difference interpolation formula for unequal intervals, we have

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

$$\therefore f(8) = f(4) + (8-4)\,52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7).1$$
$$= 48 + 400 = 448 \,//$$

and
$$f(15) = f(4) + (15-4)\times 52 + (15-4)(15-5)\times 15 + (15-4)(15-5)(15-7)$$
$$= 48 + 3102 = 3150 \,//$$