Unit - 3

Second - Order Linear Equations:

A second-order differential equation in the (unknown) function $y(x)$ is one of the form

$$G(x, y, y', y'') = 0 \quad \text{---(1)}$$

This differential equation is said to be linear provided that G is linear in the dependent variable y and its derivatives y' and y''. Thus a linear second-order equation takes the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) \quad \text{---(2)}$$

Unless otherwise noted, we will always assume that the (known) coefficients functions $A(x)$, $B(x)$, $C(x)$ and $F(x)$ are continuous on some open interval I (perhaps unbounded) on which we wish to solve this differential equation, but we donot require that they be linear functions of x. Thus the differential equation

$$e^x y'' + (\cos x) y' + (1+\sqrt{x}) y = \tan^{-1} x$$ is

linear because the dependent variable y and its derivatives y' and y'' appear linearly. By contrast, the equations $y'' = yy'$ and $y'' + 3(y')^2 + 4y^3 = 0$ are not linear because products and powers of y or its derivatives appear.

If the function $F(x)$ on the right-hand side of eqn (2) vanishes identically on I, then we call eqn (2) a homogeneous linear equation; otherwise, it is a nonhomogeneous. For example, the second-order equation

$$x^2 y'' + 2xy' + 3y = \cos x$$ is nonhomogeneous;

its associated homogeneous equation is

$$x^2 y'' + 2xy' + 3y = 0$$

In general, the homogeneous linear equation associated with eqn (2) is

$$A(x)y'' + B(x)y' + C(x)y = 0 \quad \text{---} (3)$$

* Homogeneous Second-order Linear Equations:

Consider the general second-order equation

$$A(x)y'' + B(x)y' + C(x)y = F(x) \quad \text{----} (4)$$

where the co-efficient functions A, B, C and F are continuous on the open interval I. Here, we assume in addition that $A(x) \neq 0$ at each point of I, so we can divide each term in eqn (4) by $A(x)$ and write in the form

$$y'' + P(x)y' + q(x)y = f(x) \quad \text{---} (5)$$

We will discuss first the associated homogeneous equation

$$y'' + P(x)y' + q(x)y = 0 \quad \text{----} (6)$$

A particularly useful property of this homogeneous linear equation is the fact that the sum of any two solutions of eqn (6) is again a solution, as is any constant multiple of a solution. This is the central idea of the following theorem.

Theorem: Principle of superposition for homogeneous equations:

Let y_1 and y_2 be two solutions of the homogeneous linear equation in (6) on the interval I. If c_1 and c_2 are continuous, then the linear combination

$$y = c_1 y_1 + c_2 y_2 \quad \cdots\cdots (7)$$

is also a solution of eqn (6) on I.

Proof: The conclusion follows almost immediately from the linearity of the operation of differentiation, which gives

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''$$

Then

$$y'' + py' + qy = c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2)$$

$$= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)$$

$$= c_1 . 0 + c_2 . 0$$

$$= 0$$

[because y_1 and y_2 are solutions]

Thus $y = c_1 y_1 + c_2 y_2$ is also a solution.