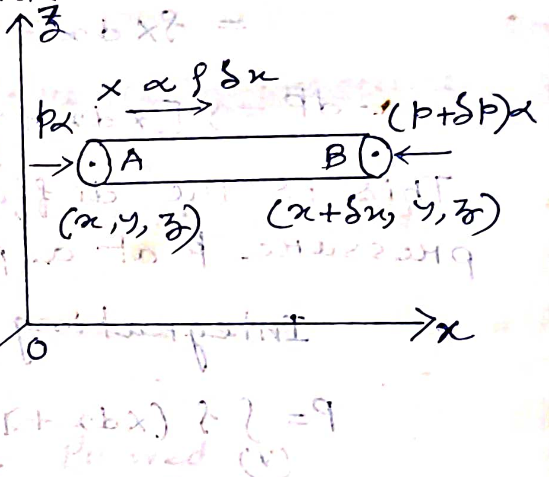
Unit - 1

2019 Equation of pressure:-

A mass of fluid is at rest under the action of given forces; to obtain the eqn which determines the pressure at any pt. of the fluid.

(06, 00, 96, 09) M

As question:- A mass of fluid is at rest under the action of given forces. obtain in rectangular cartesian co-ordinates, the eqn which determines the pressure at any point.



Let $A(x, y, z)$ be a pt. in the fluid, referred to rectangular axes.

Let $B(x+\delta x, y, z)$ be another point very near to A, such that AB is parallel to x-axis.

Let us construct a small cylinder about AB as axis with its plane ends ⊥r to AB.

Let α be the area of the plane ends of the cylinder.

Also, let p and $p+\delta p$ be the pressure at A and B, respectively.

Therefore thrust at the plane end at A is $p\alpha$ and that at B is $(p+\delta p)\alpha$.

Let ρ be the mean density of the ~~liquid~~ fluid in the cylinder AB, then mass of the cylinder $= \rho\alpha\delta x$.

Now, let X, Y, Z be the components of given forces along x-, y-, z-axes respectively per unit mass, then $X\alpha\rho\delta x$ is the force on the cylinder ∥ to the x-axis.

Thus the cylinder is in equilibrium under the forces

i) $p\alpha$ along AB ii) $X\alpha\rho\delta x$ along AB, and
iii) $(p+\delta p)\alpha$ along BA.

$\therefore$ For the equilibrium of the cylinder, we have

$(p+\delta p)\alpha = p\alpha + X\alpha\rho\delta x$

or $\delta p = \rho X \delta x$ or $\frac{\delta p}{\delta x} = \rho X$.

Taking limit as δx tends to zero, we get

$$\frac{\partial p}{\partial x} = \rho X.$$

lly, we have $\frac{\partial p}{\partial y} = \rho Y$ and $\frac{\partial p}{\partial z} = \rho Z$

Now, $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz$ [$\because$ p is a function of x, y, z]

$= \rho X dx + \rho Y dy + \rho Z dz$

$\therefore dp = \rho [X dx + Y dy + Z dz]$

This is the differential eqn determining pressure p at a point (x, y, z).

2015

★ Necessary and sufficient conditions for equilibrium of a fluid :-

08

Ex) State and establish the NASC that must be satisfied by a given system of forces X, Y, Z acting per unit mass of a fluid and ∥ to the axes of a rectangular cartesian co-ordinate system so that the fluid may maintain equilibrium.

statement :- The NASC for a fluid may maintain equilibrium, which must be satisfied under a given system of forces X, Y, Z acting per unit mass of the fluid and ∥ to the rectangular

Co-ordinate axes is

$$X\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+Y\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+Z\left(\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=0$$

Proof:- Necessary part:-

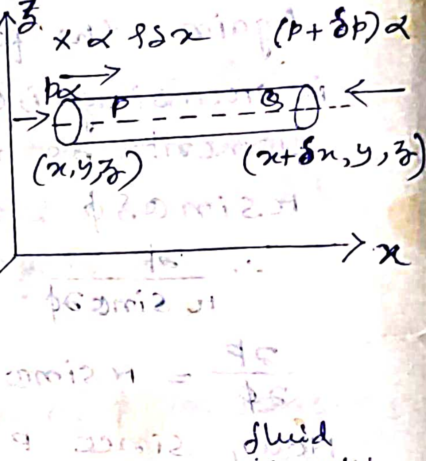
Let us take x-axis and y-axis be horizontal and z-axis vertically upward.

Let the fluid be in equilibrium.

Let $P(x,y,z)$ be a pt. in the fluid.

Let $Q(x+\delta x, y, z)$ be another pt. very near to P such that $PQ \parallel x$-axis.

Construct a small cylinder about PQ as axis with its plane ends $\perp^r$ to PQ.



Let α be the area of the plane ends of the cylinder. Also let p and $(p+\delta p)$ be the pressure at P and Q respectively.

Therefore force thrust at the plane end at P is $p\alpha$, along PQ and that at Q is $(p+\delta p)\alpha$, along QP.

Now, let ρ be the mean density of the ~~liquid~~ fluid in the cylinder PQ, then mass of the cylinder $= \rho\alpha\delta x$.

Let X, Y, Z be the components of given forces per unit mass along x-, y-, z- axes respectively, then the force on the cylinder $\parallel$ to x-axis is $X\alpha\rho\delta x$, along PQ. For equilibrium,

$$(p+\delta p)\alpha = p\alpha + X\alpha\rho\delta x$$

$$\Rightarrow \delta p = \rho X \delta x$$

$$\Rightarrow \frac{\delta p}{\delta x} = \rho X$$

Taking limit as $\delta x \to 0$, we get

$$\frac{\partial p}{\partial x} = \rho X$$

lly, we have

$$\left.\begin{aligned}\frac{\partial p}{\partial y} &= \rho Y\\ \frac{\partial p}{\partial z} &= \rho Z\end{aligned}\right\} \to (A)$$

we assume that p is continuously differentiable.

$$\therefore \frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}, \quad \frac{\partial^2 p}{\partial z \partial x} = \frac{\partial^2 p}{\partial x \partial z}, \quad \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$$

Now, $\frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y)$$

$$\Rightarrow Z\frac{\partial \rho}{\partial y} + \rho\frac{\partial Z}{\partial y} = Y\frac{\partial \rho}{\partial z} + \rho\frac{\partial Y}{\partial z}$$

$$\Rightarrow \rho\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) = Z\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial z} \rightarrow (1)$$

lly;
$$\frac{\partial^2 p}{\partial z \partial x} = \frac{\partial^2 p}{\partial x \partial z} \Rightarrow \rho\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) = X\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial x} \rightarrow (2)$$

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x} \Rightarrow \rho\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = Y\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial y} \rightarrow (3)$$

Now, multipling (1), (2), (3) by X, Y, Z and adding, we get

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

Hence the condition is necessary.

<u>Sufficient condition</u>:-

Let $X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0 \rightarrow (4)$

Now,

$$\frac{\partial}{\partial z}(\rho Y) - \frac{\partial}{\partial y}(\rho Z) = \rho\frac{\partial Y}{\partial z} + Y\frac{\partial \rho}{\partial z} - \rho\frac{\partial Z}{\partial y} - Z\frac{\partial \rho}{\partial y} \rightarrow (5)$$

$$\frac{\partial}{\partial x}(\rho Z) - \frac{\partial}{\partial z}(\rho X) = \rho\frac{\partial Z}{\partial x} + Z\frac{\partial \rho}{\partial x} - \rho\frac{\partial X}{\partial z} - X\frac{\partial \rho}{\partial z} \rightarrow (6)$$

$$\frac{\partial}{\partial y}(\rho X) - \frac{\partial}{\partial x}(\rho Y) = \rho\frac{\partial X}{\partial y} + X\frac{\partial \rho}{\partial y} - \rho\frac{\partial Y}{\partial x} - Y\frac{\partial \rho}{\partial x} \rightarrow (7)$$

Now, multipling (5), (6), (7) by $\rho X, \rho Y, \rho Z$, respectivelly and adding, we get

$$\rho X\left[\frac{\partial}{\partial z}(\rho Y) - \frac{\partial}{\partial y}(\rho Z)\right] + \rho Y\left[\frac{\partial}{\partial x}(\rho Z) - \frac{\partial}{\partial z}(\rho X)\right] + \rho Z\left[\frac{\partial}{\partial y}(\rho X) - \frac{\partial}{\partial x}(\rho Y)\right]$$

$$= \rho^2\left[X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)\right]$$
$$+ \rho\left[X\left(Y\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial y}\right) + Y\left(Z\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial z}\right) + Z\left(X\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial x}\right)\right]$$

$$= \rho^2 0 + \rho . 0$$
$$= 0$$

Thus, ~~the condition that~~

$(\rho X)dx + (\rho Y)dy + (\rho Z)dz$ is a perfect differential of some function.

$\therefore \rho X dx + \rho Y dy + \rho Z dz = dp$ (say)

$\Rightarrow \rho(X dx + Y dy + Z dz) = dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz$

$\Rightarrow \frac{\partial p}{\partial x} = \rho X, \ \frac{\partial p}{\partial y} = \rho Y, \ \frac{\partial p}{\partial z} = \rho Z$

But these are clearly the eqns of equilibrium of a fluid. Thus if ④ holds, the eqns of equilibrium follow.

Hence the condition is sufficient.

2018 (89, 94) M

Ex) Deduce that for a homogeneous fluid at rest under gravity, the free surface is horizontal.

Soln:- We have the pressure eqn is

$$dp = \rho(X\,dx + Y\,dy + Z\,dz) \to (i)$$

where X, Y are horizontal forces and Z is the vertically upward force. Here $X = 0 = Y$ and $Z = -g$. Hence (i) becomes

$$dp = -\rho g\,dz$$

Integrating, we get

$$p = k - \rho g z \quad (\because \rho \text{ is const. because the fluid is homogeneous}) \to (ii)$$

Let $p = p_0$ (const) at the origin, (i.e. $z = 0$)

$\therefore p_0 = k$

$\therefore p = p_0 - \rho g z \rightarrow$ (iii)

At the free surface, $p = 0$

eq^n (iii) becomes, $p_0 = \rho g z$

$$\Rightarrow z = \frac{p_0}{\rho g} = h \text{ (say)}$$

which is the horizontal plane.

Homogeneous liquids:-

We know that the pressure at a pt. is given by,

$$dp = \rho(X\,dx + Y\,dy + Z\,dz) \rightarrow ①$$

where ρ is the density of the ~~fluid~~ liquid at the pt. and X, Y, Z are the component forces per unit mass.

Now, if the liquid is homogeneous, then ρ is constant, thus from ①, we get

$$X.dx + Y\,dy + Z\,dz = \frac{1}{\rho}.dp = d\left(\frac{p}{\rho}\right) \rightarrow ②$$

i.e $X\,dx + Y\,dy + Z\,dz$ is a perfect differential. But if $X\,dx + Y\,dy + Z\,dz$ is a perfect differential, then the system of forces is said to be conservative. Therefore a homogeneous liquid will be in equilibrium only when the system of forces is conservative. Thus in this case, we may write,

$$X.dx + Y\,dy + Z\,dz = -dV;\ V \text{ being potential function.}$$

or $\frac{dp}{\rho} = -dV$ from ②

Integrating, we get

$$\frac{p}{\rho} + V = c;\ c \text{ being constant of integration.}$$

Heterogeneous fluids:-

In a heterogeneous ~~fluid~~ liquid ρ varies, i.e ρ is a function of independent variables x, y, z. In this case the system of forces X, Y, Z may maintain equilibrium if

$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y),\ \frac{\partial}{\partial z}(\rho X) = \frac{\partial}{\partial x}(\rho Z),$$

$$\frac{\partial}{\partial x}(\rho Y) = \frac{\partial}{\partial y}(\rho X),$$ ~~from (A) in NASC~~.

Surfaces of equal pressure:-

For a liquid at rest in equilibrium, the pressure at any pt. is given by,

$$dp = \rho(X\,dx + Y\,dy + Z\,dz)$$

Integrating, we get

$$p = \phi(x, y, z) \text{ (say)}$$

If p = constant = c (say), then we have a surface.

$$\phi(x, y, z) = c \rightarrow (1)$$

at every pt. of which the pressure is constant and equal to c.

Such a surface is called surface of equal pressure.

For different values of c, we obtain different surfaces, at every pt. of these surfaces the pressure is same.

Thus as c takes different values, (1) represents surfaces of equal pressure.

The external surface, or surface is obtained by making p equal to the pressure extend to the ~~fluid~~ liquid. If external pressure is zero, the free surface is given by

$$\phi(x, y, z) = 0$$

(02) M

Line of force:- A line of force is a curve (i.e. line) such that the tangent at every pt. of which is in the direction of the resultant force at that point.

(95; 00) M

As question:- Define a surface of equal pressure and a line of force for a fluid in equilibrium under a given system of forces.

Curves of equal pressure and density :-

2016 If a fluid is at rest under the forces X, Y, Z per unit mass, then to find the differential eqns of the curves of equal pressure and equal density.

(07,02) M

As question:-

obtain the differential eqns of the curves of equal pressure and density in the case of the fluid which is acted on by forces X, Y, Z per unit mass ‖ to the rectangular axes.

Soln:- Let the fluid be heterogeneous (so that the density is a function of x, y, z) and incompressible.

Then the surfaces of equal pressure are given by,

$$p = \text{constant}$$

i.e $dp = 0$

$$\text{or } \rho(X\,dx + Y\,dy + Z\,dz) = 0 \rightarrow ①$$

And surfaces of equal density are given by,

$$\rho = \text{constant}$$

$$\text{or } \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz = 0 \rightarrow ②$$

The curves of intersection of ① and ② are the curves of equal pressure and density.

Solving ① and ②, we get

$$\frac{dx}{Y\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial y}} = \frac{dy}{Z\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial z}} = \frac{dz}{X\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial x}} \rightarrow ③$$

We assume that p is continuously differentiable

$$\therefore \frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y}, \quad \frac{\partial^2 p}{\partial z \partial x} = \frac{\partial^2 p}{\partial x \partial z}, \quad \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$$

$$\text{Now, } \frac{\partial^2 p}{\partial y \partial z} = \frac{\partial^2 p}{\partial z \partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y)$$

$$\Rightarrow Z\frac{\partial \rho}{\partial y} + \rho\frac{\partial Z}{\partial y} = Y\frac{\partial \rho}{\partial z} + \rho\frac{\partial Y}{\partial z}$$

$$\Rightarrow Y\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial y} = -\rho\left[\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right] \rightarrow ④$$

Ily, $Z\frac{\partial p}{\partial x} - X\frac{\partial p}{\partial z} = -\rho\left[\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right] \rightarrow (5)$

and $X\frac{\partial p}{\partial y} - Y\frac{\partial p}{\partial x} = -\rho\left[\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right] \rightarrow (6)$

Putting, (4), (5), (6) in (3), we get

$$\frac{dx}{-\rho\left[\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right]} = \frac{dy}{-\rho\left[\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right]} = \frac{\partial z}{-\rho\left[\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right]}$$

$$\Rightarrow \frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

which are the differential eqns of the curves of equal pressure and density.

(06)

2017 Ex) A liquid of given volume V is at rest under the forces, $X = -\frac{\mu x}{a^2}$, $Y = -\frac{\mu y}{b^2}$, $Z = -\frac{\mu z}{c^2}$

Find the pressure at any pt. of the liquid and the surface of equal pressure.

Soln:- The pressure eqn is,

$$dp = \rho(X\,dx + Y\,dy + Z\,dz) \rightarrow (i)$$

$$= \rho\left(-\mu\frac{x}{a^2}dx - \mu\frac{y}{b^2}dy - \mu\frac{z}{c^2}dz\right)$$

$$= -\rho\mu\left(\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz\right)$$

Integrating;

$$p = -\rho\mu\left[\frac{x^2}{2a^2} + \frac{y^2}{2b^2} + \frac{z^2}{2c^2}\right] + c_1,\ c_1 - \text{arbitrary constant.}$$

$$\Rightarrow p = -\frac{\rho}{2}\mu\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] + c_1 \rightarrow (ii)$$

Now, on the free surface $p=0$,

$$\Rightarrow 0 = -\frac{\rho}{2}\mu\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]+c_1$$

$$\Rightarrow c_1 = \frac{1}{2}\rho\mu\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]$$

$$\Rightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=\frac{2c_1}{\rho\mu}=\lambda^2 \text{ (say)} \rightarrow \text{(iii)}$$

$$\Rightarrow \left(\frac{x}{a\lambda}\right)^2+\left(\frac{y}{b\lambda}\right)^2+\left(\frac{z}{c\lambda}\right)^2=1$$

which clearly represents an ellipsoid. But we know that the volume V of the ellipsoid is given by;

$$V=\frac{4}{3}\pi(a\lambda)(b\lambda)(c\lambda) \rightarrow$$ by the question, V is constant (given)

$$\Rightarrow \frac{3V}{4\pi}=abc\lambda^3$$

$$\Rightarrow \lambda^3=\frac{3V}{4abc\pi}$$

$$\Rightarrow \lambda=\left[\frac{3V}{4\pi abc}\right]^{1/3}$$

$$\therefore \text{(iii)} \Rightarrow \frac{2c_1}{\rho\mu}=\lambda^2=\left[\frac{3}{4\pi}\cdot\frac{V}{abc}\right]^{2/3}$$

$$\Rightarrow c_1=\frac{1}{2}\rho\mu\left[\frac{3V}{4\pi abc}\right]^{2/3}$$

$$\therefore \text{(ii)} \Rightarrow p=-\frac{1}{2}\rho\mu\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]+\frac{1}{2}\rho\mu\left[\frac{3V}{4\pi abc}\right]^{2/3}$$

$$=\frac{1}{2}\rho\mu\left[\frac{3V}{4\pi abc}\right]^{2/3}-\frac{1}{2}\rho\mu\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]$$

This gives pressure at any point.

Again, the equipressure surface is given by,

$$dp=0$$

$$\Rightarrow \rho[X\,dx+Y\,dy+Z\,dz]=0$$

$$\Rightarrow \frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0$$

Integrating $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k$, where k is an arbitrary constant which represents a family of surfaces of equal pressure.

(09) M

2015 * Ex) Prove that if the forces per unit mass at (x,y,z) parallel to the axes are $y(a-z)$, $x(a-z)$, xy the surfaces of equal pressure are hyperbolic paraboloids and the curves of equal pressure and density are rectangular hyperbolas.

Soln:- As given, we have

$$X = y(a-z),\ Y = x(a-z),\ Z = xy$$

so that; $\frac{\partial X}{\partial y} = a-z,\ \frac{\partial X}{\partial z} = -y,\ \frac{\partial Y}{\partial x} = a-z$

$\frac{\partial Y}{\partial z} = -x,\ \frac{\partial Z}{\partial x} = y$ and $\frac{\partial Z}{\partial y} = x.$

The condition of equilibrium is,

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

i.e $y(a-z)(-x-x) + x(a-z)(y+y) + xy[a-z-(a-z)] = 0$

which is clearly satisfied.

Now, surface of equal pressure are given by $p =$ constant i.e $dp = 0$ or

$$X\,dx + Y\,dy + Z\,dz = 0$$

or $y(a-z)dx + x(a-z)dy + xy\,dz = 0$

Putting values of X, Y, Z

Dividing by $xy(a-z)$, the surfaces of equal pressure are given by,

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{a-z} = 0$$

Integrating $\log x + \log y - \log(a-z) = \log c$ or

$\frac{xy}{a-z} = c$ which are clearly hyperbolic paraboloids.

Again the curves of equal pressure and equal density are given by,

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

or $\frac{dx}{-x-x} = \frac{dy}{y+y} = \frac{dz}{(a-z)-(a-z)}$ or

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{0}$$

The last fraction gives $dz = 0$ or $z =$ constant.

And the first two fraction are

$$\frac{dx}{x} + \frac{dy}{y} = 0 \text{ or } xy = \text{constant}.$$

Thus the curves of equal pressure and equal density are given by

$$xy = \text{constant}, \quad z = \text{constant},$$

which are clearly rectangular hyperbolas.

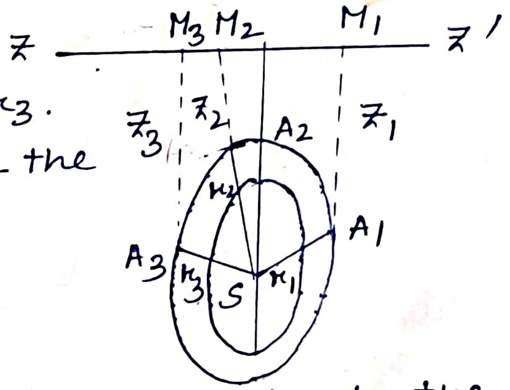
#

(09) M

2017 Ex). A closed tube in the form of an ellipse with its major axis vertical, is filled with three different liquids of densities ρ_1, ρ_2 and ρ_3 respectivelly. If the distances of the surfaces of seperation from either focus be r_1, r_2, r_3 respectivelly. Prove that

$$r_1(\rho_2-\rho_3)+r_2(\rho_3-\rho_1)+r_3(\rho_1-\rho_2)=0$$

Soln:- Let S be the focus and zz' be the directrix of the ellipse. Let A_1A_2, A_2A_3, A_3A_1 be the liquids of densities ρ_3, ρ_1, ρ_2 respectivelly.



Let $SA_1 = r_1$, $SA_2 = r_2$, $SA_3 = r_3$.

Let A_1M_1, A_2M_2, A_3M_3 be the $\perp^{rs}$ on the directrix zz'.

Let $A_1M_1 = z_1$, $A_2M_2 = z_2$,

$A_3M_3 = z_3$.

From the properties of ellipse, we know that the distance of any pt. on the ellipse from the focus = ex the $\perp^r$ distance of the pt. from the directrix where e is the eccentricity of the ellipse.

Hence $r_1 = ez_1$, $r_2 = ez_2$, $r_3 = ez_3$

$\Rightarrow z_1 = \frac{r_1}{e}$ $\quad \Rightarrow z_2 = \frac{r_2}{e}$ $\quad \Rightarrow z_3 = \frac{r_3}{e}$

Let the pressure at the pt. of seperation of the liquids A_1, A_2, A_3, be p_1, p_2, p_3 respectivelly.

Then $p_2 - p_1 = \rho_3 g (z_2 - z_1)$

$$p_3 - p_2 = \rho_1 g (z_3 - z_2)$$

$$p_1 - p_3 = \rho_2 g (z_1 - z_3)$$

adding all these and cancelling g, we get

$$0 = \rho_3 (z_2 - z_1) + \rho_1 (z_3 - z_2) + \rho_2 (z_1 - z_3)$$

$$= \rho_3 \left(\frac{r_2}{e} - \frac{r_1}{e}\right) + \rho_1 \left(\frac{r_3}{e} - \frac{r_2}{e}\right) + \rho_2 \left(\frac{r_1}{e} - \frac{r_3}{e}\right)$$

$$\Rightarrow (r_2 - r_1)\rho_3 + (r_3 - r_2)\rho_1 + (r_1 - r_3)\rho_2 = 0$$

$$\Rightarrow r_2\rho_3 - r_1\rho_3 + r_3\rho_1 - r_2\rho_1 + r_1\rho_2 - r_3\rho_2 = 0$$

$$\Rightarrow r_1(\rho_2 - \rho_3) + r_2(\rho_3 - \rho_1) + r_3(\rho_1 - \rho_2) = 0$$

Proved

2017

Example 3. *If $X = y(y+z)$, $Y = z(z+x)$, $Z = y(y-x)$, surfaces of equal pressure are the hyperbolic paraboloids $y(x+z) = c(y+z)$ and the curves of equal pressure and density are given by $y(x+z) =$ const., $(y+z) =$ const.*

Sol. We know that

$$dp = \rho[X\,dx + Y\,dy + Z\,dz].$$

Now for surfaces of equal pressure, $p =$ const., *i.e.*, $dp = 0$.
Thus surfaces of equal pressure are given by $dp = 0$

$$X\,dx + Y\,dy + Z\,dz = 0$$

or
$$y(y+z)\,dx + z(z+x)\,dy + y(y-x)\,dz = 0$$

or
$$\frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{(y-x)\,dz}{(y+z)(z+x)} = 0$$

or
$$\frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{(y+z)-(x+z)}{(y+z)(z+x)}\,dz = 0,$$

as $y - x = (y+z) - (x+z)$

or
$$\frac{dx}{z+x} + \frac{z\,dy}{y(y+z)} + \frac{dz}{z+x} - \frac{dz}{y+z} = 0$$

or
$$\frac{dx+dz}{x+z} + \frac{z\,dy - y\,dz}{y(y+z)} = 0$$

or
$$\frac{dx+dz}{x+z} + \frac{dy(z+y) - y(dy+dz)}{y(y+z)} = 0$$

or
$$\frac{dx+dz}{x+z} + \frac{dy}{y} - \frac{dy+dz}{y+z} = 0$$

Integrating, $\log(x+z) + \log y - \log(y+z) = \log c$

or
$$y(x+z) = c(y+z),$$

which is the equation giving surfaces of equal pressure.

Again curves of equal pressure and density are given by

$$\frac{dx}{\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y}} = \frac{dy}{\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z}} = \frac{dz}{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}$$

or
$$\frac{dx}{(2z+x)-(2y-x)} = \frac{dy}{-y-y} = \frac{dz}{(2y+z)-z}$$

or
$$\frac{dx}{x+z-y} = \frac{dy}{-y} = \frac{dz}{y}. \quad \ldots(1)$$

From the last two fractions,

$$dy + dz = 0 \quad \text{or} \quad y + z = \text{const.}$$

Also using the multipliers $y, x+z, y$ respectively, we have

$$y\,dx + (x+z)\,dy + y\,dz = 0$$

or
$$y\,dx + x\,dy + z\,dy + y\,dz = 0$$

or
$$xy + yz = \text{const.} \qquad \ldots(2)$$

(1) and (2) together represent curves of equal pressure and density ●

2016 **Example 4.** *Show that the forces represented by*

$$X = \mu\,(y^2 + yz + z^2),\ Y = \mu\,(z^2 + zx + x^2),\ Z = \mu\,(x^2 + xy + y^2)$$

will keep a mass of liquid at rest, if the density $\propto \dfrac{1}{(\text{dist.})^2}$ *from the plane* $x + y + z = 0$; *and the curves of equal pressure and density will be circles.*

Sol. Let d be the distance of point (x, y, z) from the plane $x + y + z = 0$; then $d = \dfrac{x+y+z}{\sqrt{3}}$.

If ρ be the density and $\rho \propto \dfrac{1}{(\text{dist.})^2}$, then

$$\rho \propto \frac{1}{d^2} \quad \text{or} \quad \rho = \frac{\lambda}{(x+y+z)^2}, \qquad \ldots(1)$$

where λ is a const, the liquid is heterogeneous.

Now the given forces will keep the heterogeneous fluid in equilibrium, if

$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y),\ \frac{\partial}{\partial x}(\rho Z) = \frac{\partial}{\partial z}(\rho X),\ \frac{\partial}{\partial y}(\rho X), = \frac{\partial}{\partial x}(\rho Y).$$

We have to show that these conditions are satisfied by the given values of X, Y, Z when ρ is given by (1).

Now
$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial y}\left[\frac{\lambda}{(x+y+z)^2} \times \mu\,(x^2 + xy + y^2)\right]$$

$$= \lambda\mu\,\frac{(x+2y)(x+y+z)^2 - 2(x+y+z)(x^2+xy+y^2)}{(x+y+z)^4}$$

$$= \lambda\mu\,\frac{[(x+2y)(x+y+z) - 2(x^2+xy+y^2)]}{(x+y+z)^3}$$

$$= \frac{\lambda\mu}{(x+y+z)^3}[-x^2 + 2yz + xz + xy]. \qquad \ldots(2)$$

and
$$\frac{\partial}{\partial z}(\rho Y) = \frac{\partial}{\partial z}\left[\frac{\lambda}{(x+y+z)^2}\,\mu\,(z^2 + zx + x^2)\right]$$

$$= \lambda\mu\,\frac{(2z+x)(x+y+z)^2 - 2(x+y+z)(z^2+zx+x^2)}{(x+y+z)^4}$$

$$= \lambda\mu\,\frac{(2z+x)(x+y+z) - 2(z^2+zx+x^2)}{(x+y+z)^3}$$

$$= \frac{\lambda\mu}{(x+y+z)^3}[-x^2 + zx + 2yz + xy]. \qquad \ldots(3)$$

Thus from (2) and (3), we get $\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y)$.

Similarly it can be shown that

$$\frac{\partial}{\partial x}(\rho Z) = \frac{\partial}{\partial z}(\rho X) \text{ and } \frac{\partial}{\partial y}(\rho X) = \frac{\partial}{\partial x}(\rho Y).$$

Thus the given forces will keep the mass of liquid at rest if ρ is given by (1).
Again the curves of equal pressure and density are

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

or
$$\frac{dx}{(x+2z)-(x+2y)} = \frac{dy}{(2x+y)-(y+2z)} = \frac{dz}{(2y+z)-(z+2x)}$$

or
$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

$$= \frac{dx+dy+dz}{0} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

Now $\quad dx + dy + dr = 0$ gives $x + y + z = \text{const.}$ ...(4)

and $\quad x\,dx + y\,dy + z\,dz = 0$ gives $x^2 + y^2 + z^2 = \text{const.}$ ...(5)

(4) and (5) together represent the curves of equal pressure and density. These curves being the curves of intersection of planes and spheres represent circles.

which ar...

2016 **Example 5.** *A fluid rests in equilibrium in a field of force, $X = y^2 + z^2 - xy - xz$, $Y = z^2 + x^2 - zy - xy$, $Z = x^2 + y^2 - xz - yz$. Show that the curves of equal pressure and density are a set of circles.*

Sol. The differential equations of curves of equal pressure and density are

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

or
$$\frac{dx}{(2z - y) - (2y - z)} = \frac{dy}{(2x - z) - (2z - x)} = \frac{dz}{(2y - x) - (2x - y)}$$

or
$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

$$= \frac{dx + dy + dz}{0} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

Now $\quad dx + dy + dz = 0$ gives $x + y + z =$ cosnt.

and $\quad x\,dx + y\,dy + z\,dz = 0$ gives $x^2 + y^2 + z^2 =$ const.

These equations together represent a set of circles.

2015 **Example 6.** *If the components parallel to the axes of the forces acting on the element of fluid at (x, y, z) be proportional to*

$$y^2 + 2\lambda yz + z^2, z^2 + 2\mu zx + x^2, x^2 + 2\nu xy + y^2,$$

show that if equilibrium be possible, we must have

$$2\lambda = 2\mu = 2v = 1.$$

[TMBU-2006H]

Sol. Here let $X = k\,(y^2 + 2\lambda yz + z^2)$,

$$Y = k\,(z^2 + 2\mu zx + x^2),\ Z = k\,(x^2 + 2\nu xy + y^2).$$

The fluid will be at rest if

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0 \quad \ldots(1)$$

or if
$$k^2\,\Sigma\,\{(y^2 + 2\lambda yz + z^2)\,(2z + 2\mu x - 2vx - 2y)\} = 0$$

or
$$\Sigma\,\{(y^2 + 2\lambda yz + z^2)\}\,\{z - y + (\mu - v)\,x\} = 0.$$

When $2\lambda = 2\mu = 2v = 1$, it becomes

$$\Sigma\{(y^2 + yz + z^2)(z - y)\} = 0$$

or $$\Sigma(y^3 - z^3) = 0$$

or $$(y^3 - z^3) + (z^3 - z^3) + (x^3 - y^3) = 0$$

which is clearly satisfied.

Thus the condition of equilibrium is satisfied when

$$2\lambda = 2\mu = 2\nu = 1.$$

This proves the result.

***Example 14.** *A given volume of liquid is at rest on a fixed plane under the action of a force, to a fixed point in the plane, varying as the distance. Find the pressure at any point of the liquid and the whole pressure on the fixed plane.*

Sol. Let the fixed point be taken as origin. Consider a point P at a distance r from the origin. Force on it $= \mu r$ towards the centre; hence pressure at this point is given by

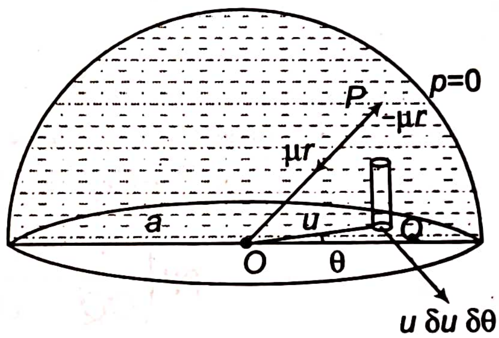


Fig. 170

$$dp = -\rho\mu\, r\, dr.$$

Integrating, $p = c - \frac{1}{2}\rho\mu r^2$.

And if $2/3\, \pi a^3$ be the given volume, the free surface is a hemisphere of radius a. Thus we have

$$p = 0 \text{ when } r = a$$

$$\therefore \qquad c = \frac{1}{2}\rho\mu a^2$$

$$\therefore \qquad p = \frac{1}{2}\rho\mu\,(a^2 - r^2).$$

This gives pressure at any point P of the liquid.

To find whole pressure on the fixed plane. To find total pressure on the fixed plane (which is here a circle of radius a) consider a point Q on the plane at a distance u from O enclosed by an element of area $u\, \delta u\, \delta\theta$.

Putting $r = u$ in (1), pressure at $Q = \frac{1}{2}\rho\mu\ (a^2 - \mu^2)$.

$\therefore$ Pressure on the element $u\ \delta u\ \delta\theta = \frac{1}{2}\rho\mu\ (a^2 - \mu^2)\ u\ \delta u\ \delta\theta$.

So total pressure $= \int_0^{2\pi}\int_0^a \frac{1}{2}\rho\mu\ (a^2 - u^2)\ u\ du\ d\theta$

$$= \pi\rho\mu\left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4\right]_0^a = \pi\rho\mu\left[\frac{1}{2}a^4 - \frac{1}{4}a^4\right] = \frac{1}{4}\pi\ \rho\mu a^4.$$

2-15 ***Example 15.** *A mass of liquid rests upon a plane subject to a central attractive force μ/r^2, situated at a distance c from the plane on the side opposite to that on which is the fluid; show that the pressure on the plane* $= \dfrac{\pi\rho\mu\ (a-c)^2}{a}$.

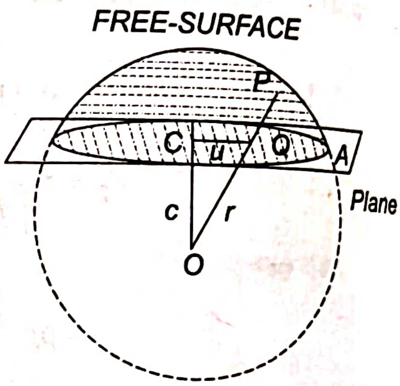


Fig. 171

Sol. The liquid rests on the plane in the form of a cap which is part of a sphere of radius a. We have to determine pressure on the plane (the circular base of the cap). Let us first consider a point P in the liquid at a distance r from the centre O of the sphere.

The only force on P is $\frac{\mu}{r^2}$ towards O. Therefore the pressure at P is given by $dp = -\rho\frac{\mu}{r^2}\,dr$.

Integrating $p = c + \dfrac{\rho\mu}{r}$

But when $r = a, p = 0$ at the free surface ; $\therefore\ c = -\dfrac{\rho\mu}{a}$

Thus $$p = \rho\mu\left(\frac{1}{r} - \frac{1}{a}\right) = \rho\mu\left(\frac{1}{OP} - \frac{1}{a}\right)$$

Now consider a point Q on the plane base of the cap ; then pressure at Q

$$= \rho\mu\left(\frac{1}{OQ} - \frac{1}{a}\right).$$

Take an element $u\ \delta u\ \delta\theta$ surrounding the point Q such that $CQ = u$ and $\angle COQ = \theta$, so that $OQ^2 = OC^2 + CQ^2 = c^2 + u^2$; then pressure on the small element $u\ \delta u\ \delta\theta$ at Q

$$\rho\mu\left(\frac{1}{OQ} - \frac{1}{a}\right) + u\ \delta u\ \delta\theta = \rho\mu\left\{\frac{1}{\sqrt{(c^2+u^2)}} - \frac{1}{a}\right\}u\ \delta u\ \delta\theta.$$

Integrating between the proper limits, so as to include the whole circular base of the cap. total pressure on the plane.

$$= \rho\mu\int_0^{\sqrt{(a^2-c^2)}}\int_0^{2\pi}\left\{\frac{1}{\sqrt{(c^2+u^2)}} - \frac{1}{a}\right\}u\ du\ d\theta,$$

[radius of the base being $\sqrt{a^2 - c^2}$]

$$= \rho\mu[\theta]_0^{2\pi}\left[\sqrt{(c^2+u^2)} - \frac{u^2}{2a}\right]_0^{\sqrt{(a^2-c^2)}}$$

$$= 2\pi\rho\mu\left(a - c - \frac{a^2 - c^2}{2a}\right)$$

$$= 2\pi\rho\mu\ (a - c)\left(1 - \frac{a + c}{2a}\right) = \frac{\pi\rho\mu}{a}(a - c)^2.$$