Elastic fluid and rotating fluid

Boyle's law:- If the temp^r remains constant, the pressure p varies inversely as the volume v i,e

$$p \propto \frac{1}{v}$$

$$\text{or } p = \frac{\lambda}{v}, \quad \lambda \text{ is a constant}$$

$$\text{or } pv = \lambda \rightarrow ⓞ$$

Charle's law:- If the pressure remains constant, the volume increases by a definite fraction of the volume at 0°c for every degree centigrade through which the temp^r increases.

Thus if v_0 and v be the volumes of the gas at 0°c and t°c, we have

$$v = v_0(1+\alpha t) \rightarrow (1)$$

where $\alpha = \frac{1}{273}$ is called the coefficient of expansion and t is temperature.

Some results for gases:- (Elastic fluid)

If temp^r remains constant, then for a gas

$$pv = \lambda \text{ (Boyle's law)}$$

Also, if ρ be the density and p the pressure, we have the mass

$$m = v\rho \text{ (constant)}$$

Now, $p = \frac{\lambda}{v} = \frac{\lambda}{m}\rho$

$$\therefore p = k\rho \rightarrow (1) \quad \text{where } \frac{\lambda}{m} = k \text{ is a constant.}$$

If the temp^r varies then by Charles law,

$$v = v_0(1+\alpha t) \rightarrow ② \quad \text{where } \alpha \text{ is the coefficient of}$$

expansion ($\alpha = \frac{1}{273}$), t is the temp^r in centigrade and v_0 is the volume at $t = 0°c$.

Now, the mass remains unchanged

i,e $\rho v = \rho_0 v_0$

$$\therefore \frac{\rho_0}{\rho} = \frac{v}{v_0} = \frac{v_0(1+\alpha t)}{v_0} = (1+\alpha t)$$

or $\rho_0 = \rho(1+\alpha t) \rightarrow (3)$

Again, if p and ρ_0 be the pressure and density at $0°c$, then $p = k\rho_0 \rightarrow (4)$

And if ρ be the density when the temp is $t°c$, where p remains unchanged, then

$$\rho_0 = \rho(1+\alpha t) \rightarrow (5)$$

Hence from (4), we get

$$p = k\rho(1+\alpha t)$$
$$= k\rho\alpha\left(t+\frac{1}{\alpha}\right)$$

$\therefore p = k\rho T$, where $T = \left(t+\frac{1}{\alpha}\right)$ and $k\alpha = k$.

T is called the absolute temp.

Elastic fluid:- $(p = k\rho)$ when temp is constant.

(06) Determination of pressure:- A mass of elastic fluid is at rest

2015 under the action of given forces. To determine the pressure at any pt.

2018 As question:- A mass of elastic fluid is at rest under the action of given forces. Determine the pressure at any pt. for both the cases when the temp remains constant and when it varies.

Let the components of forces along the direction of x-, y-, z-axes be X, Y, Z respectively per unit mass. Since the fluid is at rest under the action of the given forces, therefore the pressure eqn at any pt. is given by

$$dp = \rho(Xdx + Ydy + Zdz) \rightarrow (1)$$

Case 1:- When the temp remains constant, then

$p = k\rho \rightarrow (2)$ ~~(For elastic fluid)~~

where k is a constant and ρ the density.

Dividing (1) by (2), we get

$$\frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{k}$$

Integrating, we get

$$\log p = \frac{1}{K}\int (X\,dx + Y\,dy + Z\,dz) + \log c$$

$$\Rightarrow \log\frac{p}{c} = \frac{1}{K}\int (X\,dx + Y\,dy + Z\,dz)$$

$$\Rightarrow \frac{p}{c} = e^{\frac{1}{K}\int (X\,dx + Y\,dy + Z\,dz)}$$

$$\Rightarrow p = c\,e^{\frac{1}{K}\int (X\,dx + Y\,dy + Z\,dz)} \rightarrow (3)$$

Now, if the forces are conservative, then $X\,dx + Y\,dy + Z\,dz = -dV$, where V is the potential function.

$$\therefore (3) \Rightarrow p = c\,e^{\frac{1}{K}\int (-dV)} = c\,e^{-V/K}$$

which determines the pressure at any pt.

Case 2:- When the temp^r^ varies, then

$p = K\rho T \rightarrow (4)$ (when temp^r^ is not constant)

where K is a constant and T the absolute temp^r^.

Now, dividing (1) by (4), we get

$$\frac{dp}{p} = \frac{X\,dx + Y\,dy + Z\,dz}{KT}$$

Integrating, we get

$$\log p = \frac{1}{K}\int \frac{X\,dx + Y\,dy + Z\,dz}{T} + \log c_1 \rightarrow (5)$$

If the forces are conservative, then

$$X\,dx + Y\,dy + Z\,dz = -dV$$

$$\therefore (5) \Rightarrow \log\frac{p}{c_1} = \frac{1}{K}\int \frac{(-dV)}{T}$$

$$\Rightarrow \frac{p}{c_1} = e^{-\frac{1}{K}\int \frac{dV}{T}}$$

$$\text{or } p = c_1 e^{-\frac{1}{K}\int \frac{dV}{T}}$$

which gives the pressure at any point.

Condition of Equilibrium:- of an elastic fluid.

The necessary and sufficient condition for an elastic fluid may maintain equilibrium which must be satisfied under a given system of forces X, Y, Z acting per unit mass of the fluid and || to the rectangular co-ordinate axes is

$$X\left(\frac{dY}{dz}-\frac{\partial Z}{\partial y}\right)+Y\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+Z\left(\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=0$$

(88, 90, 92) M (statement only)

Rotating fluid:-

Eqn of pressure at a point:-

Let the liquid revolve with a uniform angular velocity ω about the vertical axis, the rotation being such that there is no relative displacement of its particles, i.e the liquid revolves like a rigid body about the axis.

Let us take the axis of rotation as the axis of z. Now we consider a particle of mass m at a distance r from the axis. This particle is moving in a circle of radius r with uniform angular velocity ω, so that the tangential acceleration due to rotation vanishes and the normal acceleration i.e $r\omega^2 (= r\dot{\theta}^2)$ acts along the inward drawn ~~nor~~ normal (i.e towards the axis being $m\omega^2 r$.

If we apply a force $m\omega^2 r$ away from the axis the force due to rotation is balanced and the system is in static equilibrium.

The components of this applied force $m\omega^2 r$ along the axes are

$m\omega^2 r\cos\theta = m\omega^2 r\cdot\frac{x}{r} = m\omega^2 x$, along x-axis.

$m\omega^2 r\sin\theta = m\omega^2 r\cdot\frac{y}{r} = m\omega^2 y$, along y-axis.

If X, Y, Z be the external forces per unit mass acting on the fluid, then

Total external force along x-axis $= m(X+\omega^2 x)$

Total external force along Y-axis $= m(Y+\omega^2 y)$

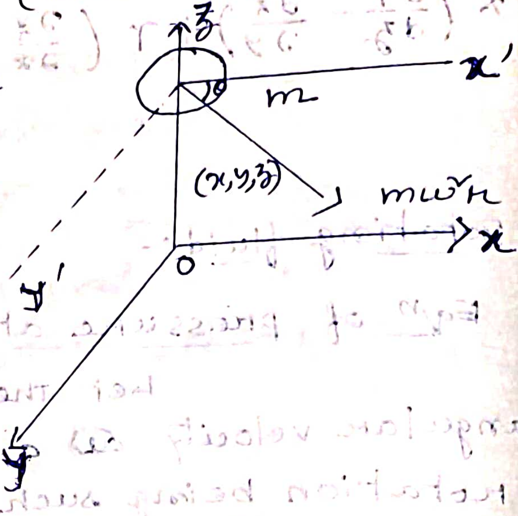
and total external force along Z-axis $= mZ$.

$\therefore$ The pressure p at a point (x, y, z) is given

$$dp = \rho[(X+\omega^2 x)dx + (Y+\omega^2 y)dy + Z\,dz]$$

$$= \rho[X\,dx + Y\,dy + Z\,dz + \omega^2(x\,dx + y\,dy)] \rightarrow (1)$$

which in general gives the pressure at a point of the rotating fluid.



If the liquid is in a closed vessel.
Therefore gravity is the only external force.

$\therefore X = 0$

$Y = 0$

$Z = -mg$

$\therefore$ (1) becomes

$$dp = \rho[\omega^2 x\,dx + \omega^2 y\,dy - g\,dz]$$

(87, 89, 91, 95)M

2016 **As question:-** A mass of homogeneous liquid contained in a vessel revolves uniformly about a vertical axis. Determine the pressure at any point and the surfaces of equal pressure.

Soln:- Since the liquid is with in a vessel. So acceleration due to gravity is the only external force, therefore due to rotation being $m\omega^2 r \perp$ to the axis of z (axis of rotation here)

Thus,

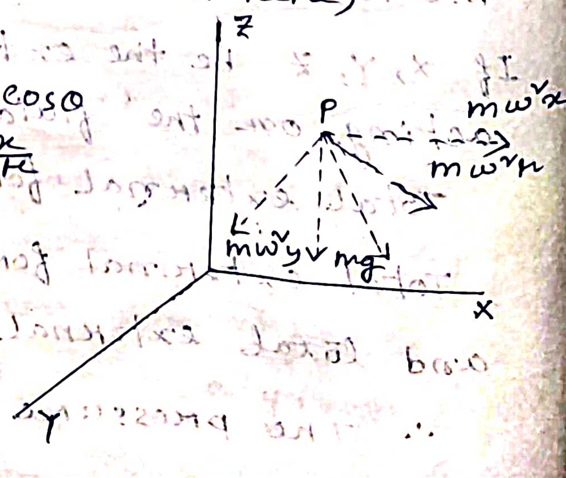
Force along x-axis $= m\omega^2 r \cdot \cos\theta$

$= m\omega^2 r \cdot \frac{x}{r}$

$= m\omega^2 x$

Force along y-axis $= m\omega^2 y$

" " z-axis $= -mg$



$\therefore$

$\therefore$ The differential eqn for pressure is

$$dp = \rho[\omega^2 x\, dx + \omega^2 y\, dy - g\, dz].$$

Integrating, we get

$$p = \rho\left[\frac{1}{2}\omega^2(x^2+y^2) - gz\right] + A \rightarrow (1)$$

This gives pressure at any point.

Now putting

$p =$ constant $= c$ (say), the surfaces of equal pressure are of the form

$$c = \rho\left[\frac{1}{2}\omega^2(x^2+y^2) - gz\right] + A$$

$$\frac{1}{2}\omega^2(x^2+y^2) - gz = \frac{c-A}{\rho} = c_1 \text{ (say)}$$

or $x^2+y^2 = \dfrac{2}{\omega^2}(gz + c_1) = \dfrac{2g}{\omega^2}\left(z + \dfrac{c_1}{g}\right)$

or $x^2+y^2 = \dfrac{2g}{\omega^2}(z - c_2)$ where $\dfrac{c_1}{g} = -c_2$

which clearly represents paraboloids of revolution of latus rectum $\dfrac{2g}{\omega^2}$.

(08) M

2017

Ex) An open vessel containing liquid is made to revolve about a vertical axis with uniform angular velocity. Find the form of the vessel and its dimensions that it may be just emptied.

Soln:- Let the liquid revolve with uniform angular velocity 'ω' then pressure at any pt. is given by,

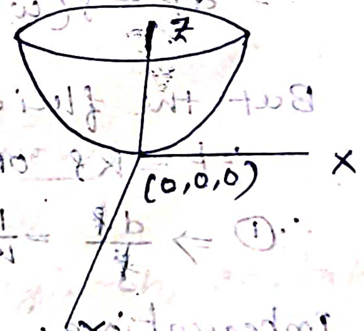
$$dp = \rho[(\omega^2 x\,dx + \omega^2 y\,dy) - g\,dz] \rightarrow (1)$$

[Since, the force along x-axis is $m\omega^2 x$ and the force along Y-axis is $m\omega^2 y$ and along z-axis, the force is only gravity which is $-g$ ($\because$ +ve z-axis taken upward).] Integrating (1), we get

$$p = \rho[\tfrac{1}{2}\omega^2(x^2+y^2) - gz] + c \rightarrow (2)$$

Let us take the lowest point of the vessel to be the origin. Then since the vessel is just emptied, there is no liquid or liquid pressure at the lowest point.

$\therefore p = 0$ when $x = y = z = 0$



i,e $c = 0$

Hence (2) becomes

$$p = \rho[\tfrac{1}{2}\omega^2(x^2+y^2) - gz]$$

Again since the vessel is just emptied, the inner surface of the vessel coincides with the free surface of the liquid. So putting $p=0$, the free surface of the liquid, i,e the inner surface of the vessel is given by

$$\tfrac{1}{2}\omega^2(x^2+y^2) - gz = 0$$

$$\Rightarrow x^2+y^2 = \frac{2g}{\omega^2} z$$

which is a paraboloid of revolution of latus rectum $\frac{2g}{\omega^2}$.

Ex) (06, 97, 92) M
2018
Imp

A mass M of elastic fluid is rotating about an axis with uniform angular velocity ω and is acted upon by an attraction towards a pt. on that axis equal to μ times the distance, μ being greater than ω^2. Prove that the eqn of a surface of equal density ρ is

$$\mu(x^2+y^2+z^2)-\omega^2(x^2+y^2)=k\log\left\{\frac{\mu(\mu-\omega^2)^2}{8\pi^3}\cdot\frac{M^2}{\rho^2K^3}\right\}$$

where z-axis is the axis of rotation and k is given by $p=k\rho$.

Soln.- Let the centre of attraction be the origin O and z-axis is vertically upward. Let us consider a pt. $P(x,y,z)$ in the fluid.

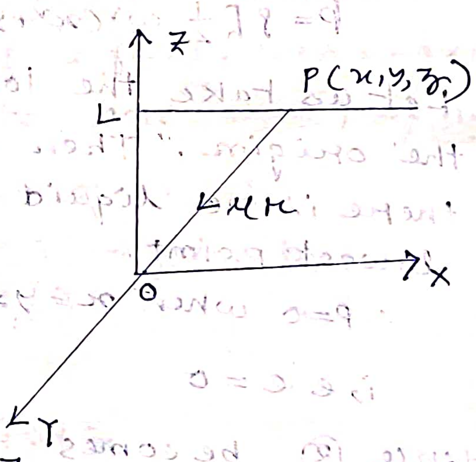
The force of attraction towards the origin $O=\mu(OP)=\mu r=\mu\sqrt{x^2+y^2+z^2}$

Let PL be the $\perp^r$ on z-axis.

The force per unit mass due to rotation is $\omega^2 PL$

$=\omega^2 u$ (say)

where $u=PL=\sqrt{x^2+y^2}$



$\therefore$ Pressure at P is given by

$$dp=\rho(\omega^2 u\,du-\mu r\,dr)\rightarrow ①$$

But the fluid is elastic

$\therefore p=k\rho$ or $dp=k\,d\rho$

$$\therefore ① \Rightarrow \frac{d\rho}{\rho}=\frac{1}{k}[\omega^2 u\,du-\mu r\,dr]$$

Integrating; $\log\rho=\frac{1}{2k}(\omega^2u^2-\mu r^2)+\log c$

$$\therefore \rho=c\,e^{(\omega^2u^2-\mu r^2)/2k}$$
$$=c\,e^{[\omega^2(x^2+y^2)-\mu(x^2+y^2+z^2)]/2k}$$

If $\rho=\rho_0$ when $x=y=z=0$, then $c=\rho_0$

$$\therefore \rho=\rho_0\,e^{-[(\mu-\omega^2)(x^2+y^2)+\mu z^2]/2k}\rightarrow ②$$

It is clear from ② that as x,y,z increase, ρ decreases, since $\mu>\omega^2$.

Now, the given liquid of mass M extends upto infinity.

$$\because M = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \rho\, dx\, dy\, dz$$

$$= 8\rho_0 \int_0^{\alpha}\int_0^{\alpha}\int_0^{\alpha} e^{-[(\mu-\omega^2)(x^2+y^2)+\mu z^2]/2k}\, dx\, dy\, dz$$

$$\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(-x)=f(x)$$

$$= 8\rho_0 \int_0^{\alpha} e^{-(\mu-\omega^2)\frac{x^2}{2k}}\, dx \times \int_0^{\alpha} e^{-(\mu-\omega^2)\frac{y^2}{2k}}\, dy$$

$$\times \int_0^{\alpha} e^{-\frac{\mu z^2}{2k}}\, dz$$

$$= 8\rho_0 \cdot \frac{\sqrt{\pi}}{2\sqrt{\frac{\mu-\omega^2}{2k}}} \cdot \frac{\sqrt{\pi}}{2\sqrt{\frac{\mu-\omega^2}{2k}}} \cdot \frac{\sqrt{\pi}}{2\sqrt{\frac{\mu}{2k}}}$$

$$\because \int_0^{\alpha} e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

or $$M = \rho_0 \frac{(2k\pi)^{3/2}}{(\mu-\omega^2)\sqrt{\mu}} \rightarrow (3)$$

Now, from (2)

$$2k \log \frac{\rho_0}{\rho} = (\mu-\omega^2)(x^2+y^2)+\mu z^2$$

or $$2k \log\left\{\frac{M(\mu-\omega^2)\sqrt{\mu}}{\rho(2k\pi)^{3/2}}\right\} = (\mu-\omega^2)(x^2+y^2)+\mu z^2$$

or $$k \log\left\{\frac{M^2(\mu-\omega^2)^2\mu}{\rho^2(2k\pi)^3}\right\} = \mu(x^2+y^2+z^2) - \omega^2(x^2+y^2)$$

or $$k \log\left\{\frac{\mu(\mu-\omega^2)^2}{8\pi^3} \cdot \frac{M^2}{\rho^2 k^3}\right\}$$

$$= \mu(x^2+y^2+z^2) - \omega^2(x^2+y^2)$$ Proved

2015

10) Ex) A mass M of a gas at uniform temperature is diffused through all space, and at each point (x, y, z) the component of force per unit mass are $-Ax, -By, -Cz$. The pressure and density at the origin are p_0 and ρ_0 resp. P.T. $ABC\rho_0 M^2 = 8\pi^3 p_0^3$.

Soln The pressure eqn is

$$dp = \rho.(X\,dx + Y\,dy + Z\,dz)$$
$$= -\rho\,(Ax\,dx + By\,dy + Cz\,dz) \longrightarrow (1)$$

Since the temparature is uniform.

So by the gas Law, $p = K\rho$

$$\Rightarrow dp = K\,d\rho \longrightarrow (2)$$

from (1) and (2) we get,

$$K\,d\rho = -\rho\,(Ax\,dx + By\,dy + Cz\,dz)$$
$$\Rightarrow \frac{d\rho}{\rho} = -\frac{1}{K}(Ax\,dx + By\,dy + Cz\,dz)$$

Integrating we get,

$$\log\rho = -\frac{1}{2K}(Ax^2 + By^2 + Cz^2) + \log c_1$$
$$\Rightarrow \rho = c_1 e^{-\frac{1}{2K}(Ax^2 + By^2 + Cz^2)} \longrightarrow (3)$$

but at the origin $x = y = z = 0$ then $\rho = \rho_0$

$\therefore (3) \Rightarrow \rho_0 = c_1$

again, $(3) \Rightarrow \rho = \rho_0 e^{-\frac{1}{2K}(Ax^2 + By^2 + Cz^2)}$

Considering the volume element $\delta x, \delta y, \delta z$ at $P(x, y, z)$

$\therefore$ mass of gas acting the volume element $= \rho\,\delta x\,\delta y\,\delta z$

$$\therefore M = \int_{x=-\alpha}^{\alpha}\int_{y=-\alpha}^{\alpha}\int_{z=-\alpha}^{\alpha} \rho\,dx\,dy\,dz$$
$$= \int_{x=-\alpha}^{\alpha}\int_{y=-\alpha}^{\alpha}\int_{z=-\alpha}^{\alpha} \rho_0\, e^{-\frac{1}{2K}(Ax^2 + By^2 + Cz^2)}\,dx\,dy\,dz$$
$$= 8\rho_0 \int_0^{\alpha} e^{-(Ax^2/2K)}\,dx \int_0^{+\alpha} e^{-(By^2/2K)}\,dy \int_0^{+\alpha} e^{-(Cz^2/2K)}\,dz$$
$$= 8\rho_0\, I_1 I_2 I_3$$

Now $I_1 = \int_0^{\alpha} e^{-(Ax^2/2K)} dx$

Let $\frac{Ax^2}{2K} = t$

$\Rightarrow \frac{2Ax\,dx}{2K} = dt$

$\Rightarrow dx = \frac{K}{Ax} dt$

$= \frac{K\,dt}{A\sqrt{\frac{2Kt}{A}}}$

$= \int_0^{\alpha} e^{-t} \frac{K\,dt}{A\sqrt{\frac{2Kt}{A}}}$

$= \sqrt{K/2A} \int_0^{\alpha} e^{-t} t^{-1/2}\,dt$

$= \sqrt{K/2A} \int_0^{\alpha} e^{-t} t^{\frac{1}{2}-1}\,dt$

$= \sqrt{K/2A}\,\sqrt{\tfrac{1}{2}}$

$= \sqrt{K/2A}\,\sqrt{\pi}$

$= \sqrt{\frac{K\pi}{2A}}$

IIly $I_2 = \sqrt{\frac{K\pi}{2B}}$

$I_3 = \sqrt{\frac{K\pi}{2C}}$

$\therefore M = 8\rho_0 \sqrt{\frac{K\pi}{2A}} \cdot \sqrt{\frac{K\pi}{2B}} \sqrt{\frac{K\pi}{2C}}$

$= 8\rho_0 \sqrt{\frac{K^3 \pi^3}{8ABC}}$

$M^2 = 8\rho_0^2 \frac{K^3 \pi^3}{ABC}$ —— (4)

Again from the gas law

$p = K\rho$

but at the origin $p = p_0$ and $\rho = \rho_0$

$\therefore p_0 = K\rho_0$

$\Rightarrow K = \frac{p_0}{\rho_0}$

(4) $\Rightarrow M^2 = 8\rho_0^2 \frac{\frac{p_0^3}{\rho_0^3}\pi^3}{ABC}$

$\Rightarrow ABC\,\rho_0 M^2 = 8\pi^3 p_0^3$

Proved

2018 ***Example 45.** *A thin sphere of radius a just filled with water rotates about a vertical diameter with angular velocity* $\omega = \sqrt{\left(\frac{2g}{3a}\right)}$; *prove that the pressure at any point of the surface of equal pressure which cuts the sphere at right angles is* $\frac{3}{4} g\rho a$, ρ *being the density of water.*

Sol. The differential equation of the pressure at any point is

$$dp = \rho\,[\omega^2 (x\,dx + y\,dy) - g\,dz].$$

Integrating,
$$p = \rho\left[\frac{1}{2}\omega^2 (x^2 + y^2) - gz\right] + c$$

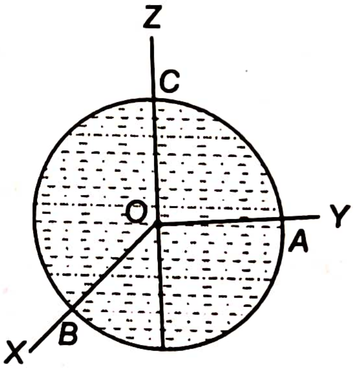


Fig. 192

The sphere is just filled with water, therefore at the highest point when

$$x = 0,\ y = 0,\ z = a,\ p = 0\ ;\ \therefore c = \rho a g$$

Thus
$$p = \rho\left[\frac{1}{2}\omega^2 (x^2 + y^2) - gz + ag\right].$$

This gives pressure at any point.

Now the surface of equal pressure, where pressure $= \frac{3}{4}\rho g a$ is given by

$$\frac{3}{4}\rho g a = \rho\left[\frac{1}{2}\cdot\frac{2g}{3a}(x^2 + y^2) - gz + ag\right]$$

or
$$x^2 + y^2 - 3gz + \frac{3}{4}a^2 = 0,$$

which is clearly a surface cutting the sphere

$$x^2 + y^2 + z^2 = a^2$$

at right angles.