UNIT-2

Fluid pressure on a plane surface:-

The pressure at every element of the surface acts along the normal to it at that pt. If the surface is plane, then all normal directions are parallel. Thus pressure acting at all elements of the plane surface are parallel and the resultant pressure of the plane is equal to the sum of these pressures.

Let the fluid be at rest under the action of any given forces and p be the pressure at a pt. (x,y). Let $\partial x \partial y$ be the elementary area enclosing the pts (x,y), then pressure on this element $= p\partial x \partial y$.

Hence total resultant pressure on the whole area

$= $ sum of all pressures at all the elements

$= \Sigma p \partial x \partial y$

$= \iint p\, dx\, dy$

the integration being taken so as to include the whole area.

In polar co-ordinates when the elementary area is taken to be $r\,d\theta\, dr$.

$\therefore$ the total pressure $= \iint p\, r\, d\theta\, dr$

(04)M(06)

Centre of pressure:-

Defn:- The centre of pressure of a plane area immersed in a fluid is the pt. at which the direction of the single force, which is equivalent to the fluid pressure on the plane surface, meets the surface.

The above definition is only for plane surfaces. The resultant fluid pressure on a curved surface is not always reducible to a single force.

It should be remembered that the depth of the centre of pressure of a plane area immersed in a fluid is greater than the depth of it's centre of gravity. However for a homogeneous plane held horizontally, the centre of pressure (c.p) coincides with C.G.

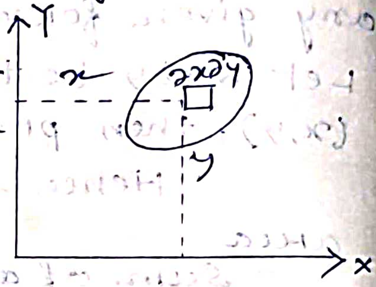
: Next

Formula for centre of pressure:-

To obtain the formula for determination of the centre of pressure of any plane area. According to question; obtain the formula for determination of the centre of pressure (C.P) of a plane area immersed in a liquid.

(88, 91, 93, 02, 06) M.

Soln.:- Let the plane area be in xy-plane. If p is the pressure per unit area at P(x,y) the pressure on the element $\partial x \partial y$ at p surrounding (x,y) is $p\partial x\partial y$.



If C $(\bar{x},\bar{y})$ are the co-ordinates of C.P, then resultant pressure $\Sigma p\partial x\partial y$ acts through $C(\bar{x},\bar{y})$, since all the direction of pressures are $\perp^r$ to the plane. Now we know that

Moment of resultant force about a pt. (or about a line) = sum of the moments of different constituent forces about that pt. (or that line)

$\therefore$ Taking moments about y-axis, we have

$$\bar{x}(\Sigma p\partial x\partial y) = \Sigma x p\partial x\partial y$$

$$\text{Or } \bar{x} = \frac{\Sigma x p\partial x\partial y}{\Sigma p\partial x\partial y} = \frac{\iint px\,dx\,dy}{\iint p\,dx\,dy}$$

the integration being taken so as to include the whole area.

Illy. taking moments about the x-axis, we get

$$\bar{y} = \frac{\iint py\,dx\,dy}{\iint p\,dx\,dy}$$; these formulas give the position of C.P. of the plane area.

Note:- When the element is taken in polar co-ordinates,

$$\bar{x} = \frac{\iint p(r\cos\theta)(r\,d\theta\,dr)}{\iint p\,(r\,d\theta\,dr)} = \frac{\iint pr^2\cos\theta\,d\theta\,dr}{\iint pr\,d\theta\,dr}$$

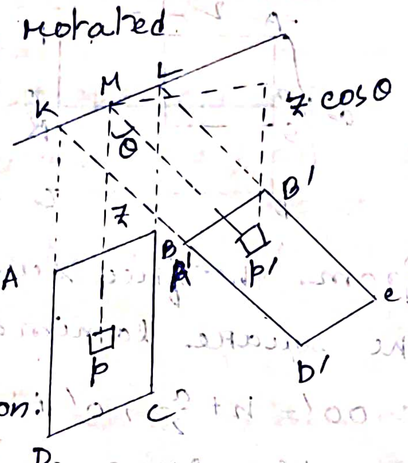
$$\text{and } \bar{y} = \frac{\iint pr\sin\theta\, r\,d\theta\,dr}{\iint p\,r\,d\theta\,dr} = \frac{\iint pr^2\sin\theta\,d\theta\,dr}{\iint pr\,d\theta\,dr}$$

Corollary:- If the fluid is at rest under gravity only, then $p = \rho g h$, where h is the depth of the elementary area below the free surface and

$$\bar{x} = \frac{\iint \rho h\, x\, dx\, dy}{\iint \rho h\, dx\, dy}; \quad \bar{y} = \frac{\iint \rho h\, y\, dx\, dy}{\iint \rho h\, dx\, dy}$$

2016 As question:- show that the position of the C.P. relative to the area remains unaltered by rotation about it's line of intersection with the effective surface.

Proof:- Let the area ABCD be rotated through an angle θ about its line of intersection KL with the free surface (i.e effective surface)



Let $dx dy$ be an element surrounding the pt. p at a depth z before rotation and p' be its new position after rotation. Then depth of p' is $z\cos\theta$. If p be the pressure at p then $p = wz$, where w is the weight per unit volume.

Now, at p', $p' = wz\cos\theta$

∴ Relative co-ordinates of the centre of pressure (c.p) in the new position are given by;

$$\bar{x} = \frac{\iint p' x\, dx\, dy}{\iint p'\, dx\, dy}$$

$$= \frac{\iint x w z \cos\theta\, dx\, dy}{\iint w z \cos\theta\, dx\, dy} = \frac{\iint x z\, dx\, dy}{\iint z\, dx\, dy}$$

and;

$$\bar{y} = \frac{\iint p' y\, dx\, dy}{\iint p'\, dx\, dy}$$

$$= \frac{\iint y . w z \cos\theta\, dx\, dy}{\iint w z \cos\theta\, dx\, dy}$$

$$= \frac{\iint y z\, dx\, dy}{\iint z\, dx\, dy}$$

∴ The value of $(\bar{x}, \bar{y})$ is independent of θ and is the same in all positions of rotation.

Hence proved

(01, 90)M

2015 Ex. A circular area of radius 'a' is immersed with its plane vertical and centre at a depth
2019 h. Find the depth of the centre of pressure.

Soln:- Let ~~ACB~~ ACB be the free surface of the liquid which is horizontal.

Let O → the centre of the circular area.

Let us take O → as the origin (pole) and the axis of x vertically downwards, through O.

Suppose this axis cuts the free surface at the point C.

$\therefore CO = h$.

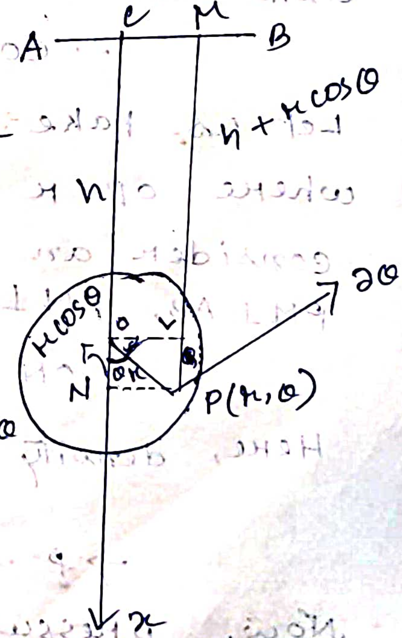
Let us take a pt. $P(r, \theta)$ on the circular area where $OP = r$ and $\angle POX = \theta$, consider an elementary area $r\,\partial\theta\,\partial r$ at P.

Let $PM \perp AB$, $PN \perp Ox$, and $OL \perp PM$.

$\therefore ON = r\cos\theta$, $PM = h + r\cos\theta$

$\therefore$ Pressure at $P = \rho g(h + r\cos\theta)$, $\rho \to$ density of the liquid.

[Pressure at P means, the pressure on the unit area at P].



$\therefore$ pressure on the elementary area $r\,\partial r\,\partial\theta$ at

$$P = \rho g(h + r\cos\theta)\, r\,\partial r\,\partial\theta$$

If the depth CP below O is $\bar{x}$,

then
$$\bar{x} = \frac{\int_{r=0}^{a}\int_{\theta=0}^{2\pi}(r\cos\theta)\rho g(h + r\cos\theta)\, r\,dr\,d\theta}{\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\rho g(h + r\cos\theta)\, r\,dr\,d\theta}$$

$$\bar{x} = \frac{2h\int_0^{\pi}\int_0^{a} r^2\cos\theta\,dr\,d\theta + 2\int_0^{\pi}\int_0^{a} r^3\cos^2\theta\,dr\,d\theta}{2h\int_0^{\pi}\int_0^{a} r\,dr\,d\theta + 2\int_0^{\pi}\int_0^{a} r^2\cos\theta\,d\theta\,dr}$$

$$= \frac{h\,\frac{r^3}{3}\Big]_0^a \sin\theta\Big]_0^{\pi} + \frac{r^4}{4}\Big]_0^a \frac{\pi}{2}}{h\,\frac{r^2}{2}\Big]_0^a\,\theta\Big]_0^{\pi} + \frac{r^3}{3}\Big]_0^a \sin\theta\Big]_0^{\pi}}$$

$= \dfrac{\pi a^4/8}{h a^2 \pi/2}$

$= \dfrac{a^2}{4h}$

$\therefore$ The depth of the C.P. below the free surface $= h + \dfrac{a^2}{4h}$

$\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

if $f(2a-x) = f(x)$

Here $\cos(2\pi - \theta) = \cos\theta$

$I = \int_0^{\pi} \cos^2\theta \, d\theta$

$= 2\int_0^{\pi/2} \cos^2\theta \, d\theta$

$= 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

$= \frac{\pi}{2}$

2018 06(a) M

Ex) Show that the c.p. of a circular area immersed in a liquid whose centre is at a depth 'h' below the surface where the density of the liquid varies as the depth is at a depth $\dfrac{2a^2 h}{a^2+4h^2}$ below the centre of the circle.

Soln:- Let ACB be the free surface of the liquid which is horizontal.

Let O be the centre of the circular area.

Let us take 'O' as the origin (pole) and the axis of x vertically downwards through O. Suppose this axis cuts the free surface at the pt. C.

$\therefore CO = h$

Let us take the pt. $P(r,\theta)$ on the circular area, where $OP = r$ and $\angle POx = \theta$.

consider an elementary area $r\,\partial\theta\,\partial r$ at P. Let $PM \perp AB$, $PN \perp Ox$ and $OL \perp PM$.

$\therefore ON = r\cos\theta$, $PM = h + r\cos\theta$

Here, density at $P \propto PM$

$= \lambda PM$

$\Rightarrow \rho = \lambda(h + r\cos\theta)$, where λ is a constant.

Now, pressure at P,

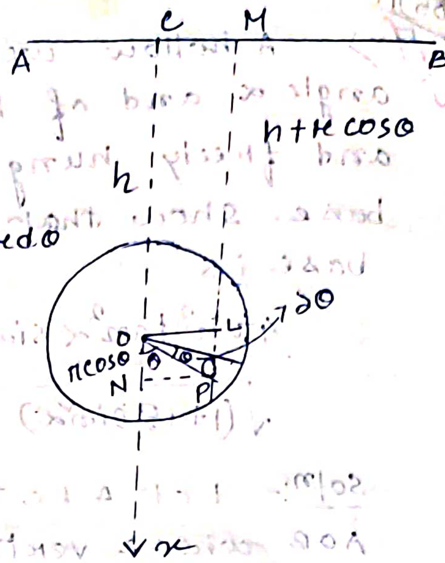
$p = \rho g\,(h + r\cos\theta)$

$= g\lambda(h + r\cos\theta)^2$

Therefore,

Depth of resultant C.P. below the centre of the circle

$$H=\frac{\int_{\theta=0}^{2\pi}\int_{r=0}^{a}(r\cos\theta)\,g\lambda\,(h+r\cos\theta)^2\,r\,dr\,d\theta}{\int_{\theta=0}^{2\pi}\int_{r=0}^{a}g\lambda\,(h+r\cos\theta)^2\,r\,d\theta\,dr}$$



$$=\frac{2g\lambda\int_{\theta=0}^{\pi}\int_{r=0}^{a}r^2\cos\theta\,(h^2+r^2\cos^2\theta+2hr\cos\theta)\,dr\,d\theta}{2g\lambda\int_{\theta=0}^{\pi}\int_{r=0}^{a}r\,(h^2+r^2\cos^2\theta+2hr\cos\theta)\,dr\,d\theta}$$

[since $\int_0^{2a}f(x)\,dx=2\int_0^{a}f(x)\,dx$ when $f(2a-x)=f(x)$]

~~Here, $f(\theta)$ is a f^n of $\cos\theta$~~

~~$\therefore f(2\pi-\theta)=f(\theta)$~~

$$=\frac{h^2\int_{r=0}^{a}r^2\,dr\int_{\theta=0}^{\pi}\cos\theta\,d\theta+\int_0^a r^4\,dr\int_0^\pi\cos^3\theta\,d\theta+2h\int_0^a r^3\,dr\int_0^\pi\cos^2\theta\,d\theta}{h^2\int_0^a r\,dr\int_0^\pi d\theta+\int_0^a r^3\,dr\int_0^\pi\cos^2\theta\,d\theta+2h\int_0^a r^2\,dr\int_0^\pi\cos\theta\,d\theta}$$

[Now, $\int_0^\pi\cos\theta\,d\theta=0$, since $f(\theta)=-f(\pi-\theta)$

lly, $\int_0^\pi\cos^3\theta\,d\theta=0$

$\int_0^\pi\cos^2\theta\,d\theta=2\int_0^{\pi/2}\cos^2\theta\,d\theta$, since if $f(\theta)=\cos^2\theta$

$=2\cdot\frac{1}{2}\cdot\frac{\pi}{2}$ then $f(\pi-\theta)=\cos^2\theta$]

$=\frac{\pi}{2}$

Therefore,

$$H=\frac{h^2\cdot\frac{a^3}{3}\cdot 0+\frac{a^5}{5}\cdot 0+2h\cdot\frac{a^4}{4}\cdot\frac{\pi}{2}}{h^2\cdot\frac{a^2}{2}\cdot\pi+\frac{a^4}{4}\cdot\frac{\pi}{2}+\frac{a^3}{3}\cdot 0}$$

$$=\frac{\frac{1}{2}\cdot a^4 h\cdot\frac{\pi}{2}}{\frac{1}{2}a^2h^2\cdot\pi+\frac{1}{8}a^4\cdot\pi}$$

$$=\frac{\frac{1}{2}a^2h}{h^2+\frac{1}{4}a^2}$$

$$=\frac{2a^2h}{a^2+4h^2}\quad\text{(shown)}$$

Note:- $I=\int_0^\pi\cos^{2x-1}\theta\,d\theta=\int_0^\pi\cos^{2x-1}(\pi-\theta)\,d\theta$

$=\int_0^\pi-\cos^{2x-1}\theta\,d\theta$

$=-I$

$\therefore 2I=0$

$\Rightarrow I=0$ #

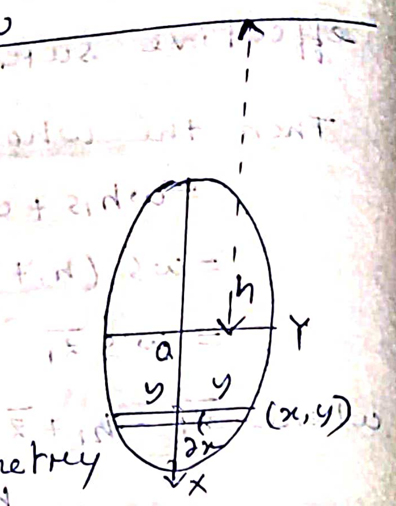
2017 (07) M

Ex) An ellipse is completely immersed with its minor axis horizontal and at a depth h; find the position of the centre of pressure.

Soln: Referred to major and minor axes as axes of reference, the eqn of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (1)$$

consider an elementary strip of breadth δx at a depth x below the centre of the ellipse.



Area of the strip $= 2y\delta x$

Depth of the strip below free surface $= h+x$

$\therefore$ pressure on the strip

$= \rho g\,(h+x)\,2y\,dx$

If $(\bar{x}, \bar{y})$ is c.p., then the symmetry of the ellipse about major axis $\bar{y}$ lies on the major axis and

$\therefore \bar{y} = 0$

$$\therefore \bar{x} = \frac{\int_{-a}^{a} x\rho g\,(h+x)2y\,dx}{\int_{-a}^{a} \rho g\,(h+x)\,2y\,dx}$$

$$= \frac{\int_{-a}^{a} x(h+x)\sqrt{(a^2-x^2)}\,dx}{\int_{-a}^{a} (h+x)\sqrt{(a^2-x^2)}\,dx}$$

as $y = \frac{b}{a}\sqrt{(a^2-x^2)}$. from ⓒ

$$= \frac{2\int_0^a x^2\sqrt{(a^2-x^2)}\,dx}{2\int_0^a h\sqrt{(a^2-x^2)}\,dx}$$; other integrals vanish being integrals of odd functions.

$$= \frac{\int_0^{\pi/2} a^2\sin^2\theta\cos^2\theta\,d\theta}{\int_0^{\pi/2} h\cos^2\theta\,d\theta}$$

$$= \frac{a^2\int_0^{\pi/2}\sin^2\theta\,(1-\sin^2\theta)\,d\theta}{h\int_0^{\pi/2}\cos^2\theta\,d\theta}$$

$$= \frac{a^2}{h}\,\frac{\int_0^{\pi/2}(\sin^2\theta - \sin^4\theta)\,d\theta}{\int_0^{\pi/2}\cos^2\theta\,d\theta}$$

$$= \frac{a^2}{h}\,\frac{\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]}{\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]}$$

$$= \frac{a^2}{h}\,\frac{\frac{\pi}{4} - \frac{3\pi}{16}}{\frac{\pi}{4}}$$

$$= \frac{a^2}{h} \frac{\frac{4-3}{16}}{\frac{1}{4}}$$

$$= \frac{a^2}{h} \frac{1/16}{1/4}$$

$$= \frac{a^2}{h} \frac{1}{16} \times 4$$

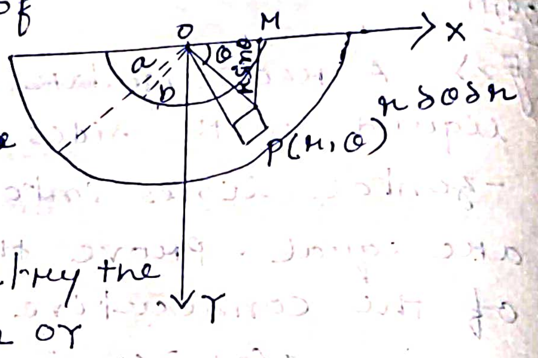
$$= \frac{a^2}{4h} \quad \#$$

2016 (08,06) M

Ex) If an area is bounded by two concentric semi circles with their common bounding diameter in the free surface, prove that the depth of the centre of pressure is

$$\frac{3\pi}{16}\left[\frac{(a+b)(a^2+b^2)}{(a^2+b^2+ab)}\right],$$ where a and b are the radii.

Soln:- Let the common centre of the circles be the origin, common diameter be the x-axis, and the y-axis be vertically downward.



Clearly due to symmetry the centre of pressure will be on OY

$\therefore \bar{x}=0$

$\therefore$ if $C(\bar{x},\bar{y})$ be the C.P. then $C(0,\bar{y})$.

Now, we consider an elementary area $r\partial\theta\partial r$ at $P(r,\theta)$ where $OP=r$ and $\angle POX=\theta$.

Depth of this element below the free surface

$= r\sin\theta \; (=MP)$.

$\therefore$ Pressure on the elementary area

$= \rho g(r\sin\theta)\, r\partial\theta\partial r$.

Let the radius of the inner circle be 'b' and that of the outer circle be 'a'.

$$\therefore \bar{y} = \frac{\int_{r=b}^{a}\int_{\theta=0}^{\pi} r\sin\theta\, \rho g r\sin\theta\, r d\theta dr}{\int_{r=b}^{a}\int_{\theta=0}^{\pi} \rho g r\sin\theta\, r d\theta dr}$$

$$= \frac{\int_b^a\int_0^\pi r^3\sin^2\theta\, d\theta dr}{\int_b^a\int_0^\pi r^2\sin\theta\, d\theta dr}$$

$$\therefore \bar{y} = \frac{\int_b^a r^3 dr \times 2\int_0^{\pi/2}\sin^2\theta\, d\theta}{\int_b^a r^2 dr \times \int_0^\pi \sin\theta\, d\theta}$$

$$= \frac{\left[\frac{r^4}{4}\right]_b^a \times 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}}{\left[\left(\frac{r^3}{3}\right)\right]_b^a\left[-\cos\theta\right]_0^\pi}$$

$$= \frac{3\pi}{8} \frac{a^4 - b^4}{(a^3 - b^3)(1+1)}$$

$$= \frac{3\pi}{16} \frac{(a^4 - b^4)}{(a^3 - b^3)}$$

$$= \frac{3\pi (a^2+b^2)(a+b)(a-b)}{16 (a-b)(a^2+b^2+ab)}$$

$$= \frac{3\pi (a+b)(a^2+b^2)}{16 (a^2+b^2+ab)}$$ **Proved**

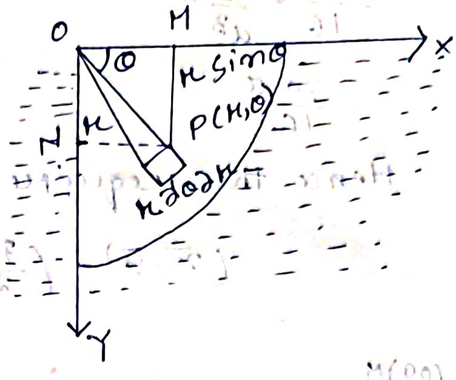
2016(08)M

Ex: A quadrant of a circle is just immersed vertically in a heavy homogeneous liquid with one edge in the surface. Determine the position of the centre of pressure.

Soln:- Let the edge in the surface be taken as the x-axis and the centre of the circle as the origin O and y-axis vertically downward.

Now, let us consider an elementary area $r\,\partial\theta\,\partial r$ at $P(r, \theta)$ where $OP = r$ and $\angle POX = \theta$.

Depth of this element below the surface is $PM = r\sin\theta$ and let $PN = r\cos\theta$ be the distance of P from the Y-axis. let ρ (constant) be the density at P.



Pressure at P per unit area is $p = \rho g h = \rho g\,[MP] = \rho g\, r\sin\theta$

pressure on the elementary area at P is $(\rho g r\sin\theta)(r\,\partial\theta\,\partial r)$

If cartesian coordinates of P are (x,y), then

$x = OM = PN = r\cos\theta$

and $y = r\sin\theta$

Now, if $(\bar{x},\bar{y})$ is the centre of pressure, then we have,

$$\bar{x} = \frac{\int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r\cos\theta\,(\rho g r\sin\theta)(r\,d\theta\,dr)}{\int_0^a\int_0^{\pi/2}(\rho g r\sin\theta)(r\,d\theta\,dr)}$$

$$= \frac{\int_0^a\int_0^{\pi/2} r^3\sin\theta\cos\theta\,dr\,d\theta}{\int_0^a\int_0^{\pi/2} r^2\sin\theta\,d\theta\,dr}$$

$$= \frac{\int_0^a r^3\,dr \times \int_0^{\pi/2}\sin\theta\,d(\sin\theta)}{\int_0^a r^2\,dr \times \int_0^{\pi/2}\sin\theta\,d\theta}$$

$$= \frac{\left[\frac{r^4}{4}\right]_0^a \times \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}}{\left[\frac{r^3}{3}\right]_0^a \times \left[-\cos\theta\right]_0^{\pi/2}}$$

$$= \frac{\frac{1}{8}a^4\times 1}{\frac{a^3}{3}\times 1}$$

$$= \frac{a^4}{8}\times\frac{3}{a^3}$$

$$= \frac{3a}{8}$$

$$\bar{y} = \frac{\int_0^a\int_0^{\pi/2}(r\sin\theta)(\rho g r\sin\theta)(r\,d\theta\,dr)}{\int_0^a\int_0^{\pi/2}(\rho g r\sin\theta)(r\,d\theta\,dr)}$$

$$= \frac{\int_0^a r^3\,dr \times \int_0^{\pi/2}\sin^2\theta\,d\theta}{\int_0^a r^2\,dr \times \int_0^{\pi/2}\sin\theta\,d\theta}$$

$$= \frac{\frac{a^4}{4}\times\frac{1}{2}\cdot\frac{\pi}{2}}{\frac{a^3}{3}}$$

$$= \frac{a^4}{16} \times \frac{3\pi}{a^3}$$

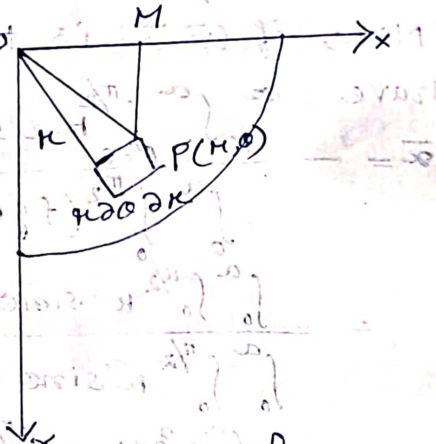
$$= \frac{3a\pi}{16}$$

Hence the required centre of pressure

$$(\bar{x}, \bar{y}) = \left(\frac{3a}{8}, \frac{3a\pi}{16}\right) \#$$

2018 2015 (09)M

Ex) A quadrant of a circle is just immersed vertically with one edge in the surface in a liquid, the density of which varies as the depth. Determine the centre of pressure.



Soln:- Let the edge in the surface be taken as the x-axis and the centre of the circle as the origin O and y-axis vertically downward.

Now, let us consider an elementary area $r\partial\theta\partial r$ at $P(r,\theta)$ where $OP = r$ and $\angle POX = \theta$.

Depth of this element below the surface $= r\sin\theta$.

Here density $\rho \propto r\sin\theta$ | let the radius of the circle be a.

or $\rho = \lambda r\sin\theta$, λ is a constant.

pressure at P per unit area is

$$p = \rho g h = \rho g [MP] = \rho g r\sin\theta$$
$$= \lambda g r^2 \sin^2\theta$$

pressure on the elementary area $r\partial\theta\partial r$ at P is

$$(\lambda g r^2 \sin^2\theta)(r\partial\theta\partial r) = \lambda g r^3 \sin^2\theta\, \partial\theta\, \partial r$$

Now, if $(\bar{x}, \bar{y})$ is the centre of pressure then, we have

$$\bar{x} = \frac{\int_0^a \int_0^{\pi/2} r\cos\theta\, \lambda g r^2 \sin^2\theta\, r\, d\theta\, dr}{\int_0^a \int_0^{\pi/2} \lambda g r^2 \sin^2\theta\, r\, d\theta\, dr}$$

$$= \frac{\int_0^a r^4 dr \times \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta}{\int_0^a r^3 dr \times \int_0^{\pi/2} \sin^2\theta\, d\theta}$$

$$= \frac{\left[\frac{r^5}{5}\right]_0^a \times \int_0^{\pi/2} d\left(\frac{\sin^3\theta}{3}\right)}{\left[\frac{r^4}{4}\right]_0^a \times \frac{1}{2}\cdot\frac{\pi}{2}}$$

$$= \frac{\frac{a^5}{5} \times \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}}{\frac{\pi a^4}{16}}$$

$$= \frac{a^5/15}{\pi a^4/16}$$

$$= \frac{16a}{15\pi}$$

and

$$\bar{y} = \frac{\int_0^a \int_0^{\pi/2} r\sin\theta\,(\lambda g r^2 \sin^2\theta)\, r\,d\theta\,dr}{\int_0^a \int_0^{\pi/2} (\lambda g r^2 \sin^2\theta)\, r\,d\theta\,dr}$$

$$= \frac{\int_0^a \int_0^{\pi/2} r^4 \sin^3\theta\, d\theta\, dr}{\int_0^a \int_0^{\pi/2} r^3 \sin^2\theta\, d\theta\, dr}$$

$$= \frac{\int_0^a r^4 dr \times \int_0^{\pi/2} \sin^3\theta\, d\theta}{\int_0^a r^3 dr \times \int_0^{\pi/2} \sin^2\theta\, d\theta}$$

$$= \frac{\left[\frac{r^5}{5}\right]_0^a \times \frac{2}{3}}{\left[\frac{r^4}{4}\right]_0^a \times \frac{1}{2}\cdot\frac{\pi}{2}}$$

$$= \frac{2a^5}{15} \Big/ \frac{a^4\pi}{16}$$

$$= \frac{32a}{15\pi}$$ #

2018 **Example 2.** *Prove that the depth of C.P of parallelogram whose two sides are horizontal and at depths h and k below the free surface of a liquid whose density varies as depth below the surface is* $\frac{3}{4}\left(\frac{h^3+h^2k+hk^2+k^3}{h^2+hk+k^2}\right)$.

Sol. Let $ABCD$ be a parallelogram whose two sides AB and DC are horizontal and be at depth h and k respectively below the free surface of the liquid. Let us divide the parallelogram into elementary strips parallel to the free surface. Considering an elementary strip PQ of width dx at a depth x below the free surface.

Let ρ be the density of liquid at the level of the strip PQ then from the question $\rho \propto x$.

i.e. $\rho = ax$ where a is a constant.

$\therefore$ The thrust on the elementary strip PQ is given by

$$dT = \text{area} \times \text{pressure at its C.G}$$

$$= l.dx \times \rho g x \qquad \text{where } AB = DC = l$$

$$= l.dx\,.ax.gx = algx^2.dx$$

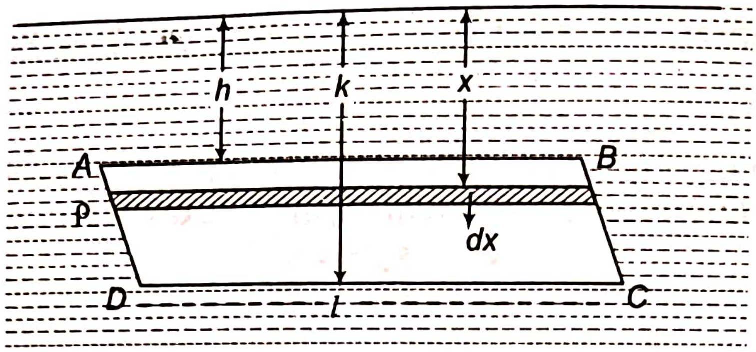


Fig. 68

Depth of C.P is given by

$$\bar{x} = \frac{\int_h^k x\, dT}{\int_h^k dT}$$

$$= \frac{\int_h^k x^3 dx}{\int_h^k x^2 dx}$$

$$= \left[\frac{\frac{x^4}{4}}{\frac{x^3}{3}}\right]_h^k$$

$$\bar{x} = \frac{3}{4}\left[\frac{k^4 - h^4}{k^3 - h^3}\right]$$

$$\bar{x} = \frac{3}{4}\left[\frac{k^3 + k^2 h + kh^2 + h^3}{k^2 + kh + h^2}\right]$$

2016 **Example 31.** *A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth ; if the minor axis be in the surface, find the eccentricity in order that the focus may be the centre of pressure.* $\left[\text{An } e = \dfrac{32}{15\pi}\right]$ **[TMBU-02H]**

Sol. Referred to major and minor axes as axes of reference, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Now consider an elementary strip of breadth δx at a depth x below the free surface, then

$\rho = \lambda x$, (area of strip) $= 2y\delta x$.

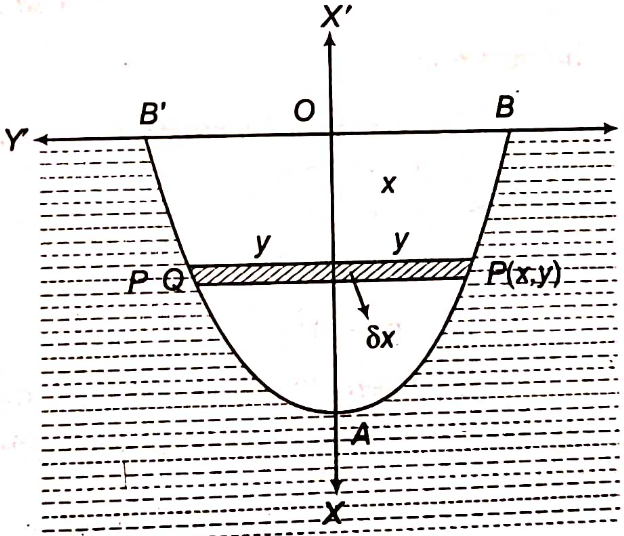


Fig. 102

∴ pressure on the elementary strip

$= \rho g x . 2y\, \delta x = \lambda g x^2 . 2y dx.$

By symmetry, C.P. lies the axis of x. $\therefore \bar{y} = 0$

and
$$\bar{x} = \frac{\int_0^a x . \lambda g x^2 . 2y\, dx}{\int_0^a \lambda g x^2 . 2y\, dx}$$

$$= \frac{\int_0^a x^3 \sqrt{(a^2 - x^2)}\, dx}{\int_0^a x^2 \sqrt{(a^2 - x^2)}\, dx}, \qquad \left[\text{from (1) } y = \frac{b}{a} x \sqrt{(a^2 - x^2)}\right]$$

$$= \frac{\int_0^{\pi/2} \sin^3\theta \cos^2\theta d\theta dx}{\int_0^a \sin^2\theta \cos^2\theta\, d\theta} \qquad (\text{where } x = a \sin\theta)$$

$$= \frac{32a}{15\pi}. \qquad (\text{by using gamma function})$$

But C.P. coincides with the focus, so

$$ae = \frac{32a}{15\pi}$$

or
$$e = \frac{32}{15\pi} < 1 \qquad \textbf{Ans.}$$