Resultant pressure on curved surface:-

We know that pressure at every element of a surface acts along the normal to the surface at that point. In case of plane surface pressure at all the elements are ∥ and their sum gives the resultant pressure on the plane surface. But in the case of a curved surface the normals at different elements have different directions in the space. The forces along them in general can not be compounded into a single force.

However, we can find the components of all the forces along three mutually ⊥r directions. For convenience, one direction is taken to be vertical and the other two horizontal.

Thus to determine pressure on a curved surface, we must determine the vertical and horizontal components of this pressure.

Resultant vertical thrust:-

To find the resultant vertical pressure on any surface of a homogeneous liquid at rest under the action of gravity.

\# Let us determine vertical resultant thrust on the curved surface PQRS.

From every point of the curve PQRS let us draw vertical lines to meet the free surface in the curve P'Q'R'S'. The vertical cylinder (PQRS, P'Q'R'S') so formed is at rest under the forces

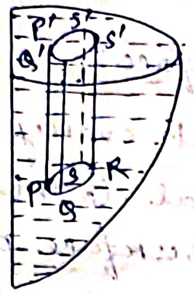
i) weight of the contained liquid.

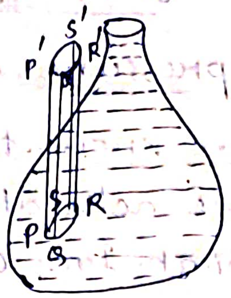
ii) vertical component of the reaction of the surface PQRS.

For equilibrium these must be equal

∴ vertical component of the thrust PQRS
= weight of the liquid in (PQRS, P'Q'R'S')
= weight of the superincumbent liquid.

(The liquid in the cylinder is called the superincumbent liquid).

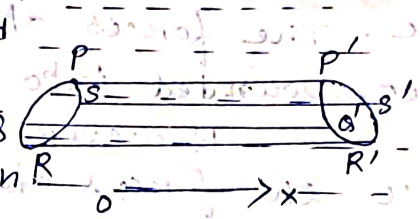




Resultant horizontal thrust:-

To find the resultant horizontal pressure in a given direction.

Let PQRS be the curved surface immersed in a fluid. We have to find horizontal thrust in the assigned direction OX.



Let us draw lines ∥ to OX, from every point of PQRS and take a vertical cross section P'Q'R'S' of the cylinder formed by these lines.

Then this cylinder (PQRS, P'Q'R'S') is in equilibrium under the following horizontal forces:

i) Horizontal thrust on PQRS along OX.

ii) Thrust on P'Q'R'S'.

For equilibrium these forces must be equal and opposite.

Hence horizontal thrust along OX = pressure on P'Q'R'S' i.e on the projection of the surface on a vertical plane ⊥r to OX.

Resultant thrust on a solid wholly or partially immersed in the liquid:-

Principle of Archimedes:-

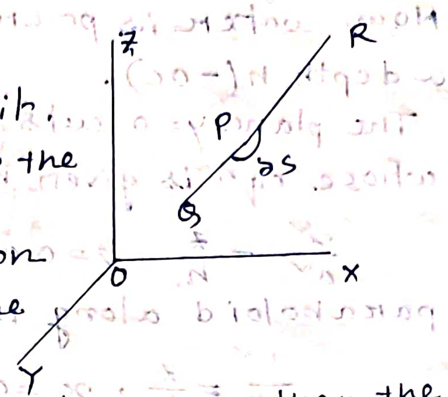
If a body is wholly or partially immersed in a liquid, the resultant thrust of the fluid on the body is equal to the weight of the liquid displaced by it and acts vertically upwards through the centre of gravity of the displaced liquid.

(07)M

2017

Ex) To obtain the resultant pressure, on any surface, of a fluid at rest under the action of any given forces.

Soln:- Let QR be the curved surface and P be any pt. on it. Also let p be the pressure at the pt. P.



If l, m, n are the direction cosines of the normal to the surface at $P(x,y,z)$ and δs an element of the surface enclosing P, then the pressure on this element is $p\delta s$ acting along the normal to the surface.

The components of the pressure on this element ‖ to axes are $lp\delta s$, $mp\delta s$, $np\delta s$.

Now in general such pressures cannot be reduced to a single force.

Let X, Y, Z and L, M, N be the resultant pressures ‖ to the axes and the resultant couples respectivelly; then

$$X = \Sigma lp\delta s = \iint lp\,ds,\quad Y = \iint mp\,ds$$

$$Z = \iint np\,ds$$

$$L = \Sigma(ynp\delta s - zmp\delta s)$$
$$= \iint (ynp\,ds - zmp\,ds)$$
$$= \iint p(ny - mz)\,ds$$

$$M = \iint p(lz - nx)\,ds,\quad N = \iint p(mx - ly)\,ds$$

the integration being such as to include the whole surface under consideration.

2016

(06, 99, 08)M

Ex) A vessel in the form of an elliptic paraboloid whose axis is vertical and eqn $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{h}$ is divided into four equal compartments by its principal planes. Into one of these water is poured to the depth h. Prove that if the resultant pressure on the curved portion is reduced to two forces, one vertical and the other horizontal, the line of action of the latter will pass through the pt. $\left(\frac{5}{16}a, \frac{5}{16}b, \frac{3}{7}h\right)$.

Soln:- The given surface is

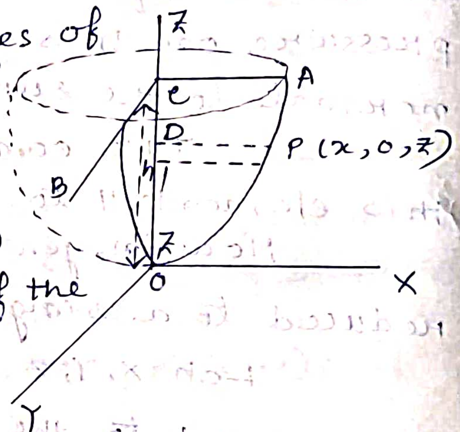
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z}{h} \to (1)$$

Now, water is poured into the portion (O, ABC) to a depth h(=OC).

The plane y=0 cuts the paraboloid in the curve OA whose eqn is given by;

$$\frac{x^2}{a^2} = \frac{z}{h}, \; y=0$$ and the plane $x=0$ cuts the paraboloid along the curve OB whose eqn is

$$\frac{y^2}{b^2} = \frac{z}{h}, \; x=0$$



If $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the c.p. of the face OAC, then considering in the plane OAC a horizontal strip (DP) of breadth δz at a height $z=(OD)$ above the vertex O. The c.p. of the strip is $M\left(\frac{x}{2},0,z\right)$ because $P(x,0,z)$.

$$\therefore \bar{x} = \frac{\int_0^h \frac{x}{2}\rho g(h-z)x\,dz}{\int_0^h \rho g(h-z)x\,dz}$$

$$= \frac{1}{2}\,\frac{\int_0^h (h-z)\frac{z a^2}{h}\,dz}{\int_0^h (h-z)a\sqrt{\frac{z}{h}}\,dz}$$

$$= \frac{a}{2\sqrt{h}}\,\frac{\int_0^h (hz-z^2)\,dz}{\int_0^h (hz^{1/2}-z^{3/2})\,dz}$$

$$\bar{x} = \frac{a}{2\sqrt{h}}\,\frac{\left[h\frac{z^2}{2}\right]_0^h - \left[\frac{z^3}{3}\right]_0^h}{\left[h\cdot\frac{2}{3}\cdot z^{3/2}\right]_0^h - \left[\frac{2}{5}z^{5/2}\right]_0^h}$$

$$= \frac{a}{2\sqrt{h}}\,\frac{h^3\left(\frac{1}{2}-\frac{1}{3}\right)}{h^{3/2}\left(\frac{2}{3}-\frac{2}{5}\right)}$$

$$= \frac{ah^3}{2h^3}\,\frac{1/6}{4/15} = \frac{5a}{16}$$

$$\bar{y} = 0$$

and $$\bar{z} = \frac{\int_0^h z\rho g(h-z)x\,dz}{\int_0^h \rho g(h-z)x\,dz}$$

$$= \frac{\int_0^h (h-z)\, z a \cdot \sqrt{\frac{z}{a}}\, dz}{\int_0^h (h-z)\, a \cdot \sqrt{\frac{z}{h}}\, dz}$$

$$= \frac{\int_0^h (h z^{3/2} - z^{5/2})\, dz}{\int_0^h (h z^{1/2} - z^{3/2})\, dz}$$

$$= \frac{\left[h \frac{2}{5} z^{5/2}\right]_0^h - \left[\frac{2}{7} z^{7/2}\right]_0^h}{\left[h \frac{2}{3} z^{3/2}\right]_0^h - \left[\frac{2}{5} z^{5/2}\right]_0^h}$$

$$= \frac{h^{7/2}\left(\frac{2}{5} - \frac{2}{7}\right)}{h^{5/2}\left(\frac{2}{3} - \frac{2}{5}\right)}$$

$$= h \frac{4/35}{4/15}$$

$$= \frac{3}{7} h$$

Thus the co-ordinates of the c.p. of the face OAC are $\left(\frac{5}{16} a, 0, \frac{3}{7} h\right)$

lly, the co-ordinates of the c.p. of the face OBC are $\left(0, \frac{5}{16} b, \frac{3}{7} h\right)$.

Now, if X, Y be the horizontal forces ‖ to axes of x and y respectively, then

X = horizontal pressure along x-axis
= pressure on the projection of the curved surface on a plane ⊥r to OX.
= pressure on OBC acting through the c.p. of OBC i.e through $\left(0, \frac{5}{16} b, \frac{3}{7} h\right)$.

lly;
Y = pressure on OAC acting through the c.p. of OAC i.e through $\left(\frac{5}{16} a, 0, \frac{3}{7} h\right)$.

The horizontal lines through these pts. ‖ to axes of x and y respectively intersect at a pt. $H\left(\frac{5}{16} a, \frac{5}{16} b, \frac{3}{7} h\right)$, ~~through which therefore the resultant horizontal thrust passes~~. Again vertical thrust on the curved surface, Z
= weight of the liquid contained.

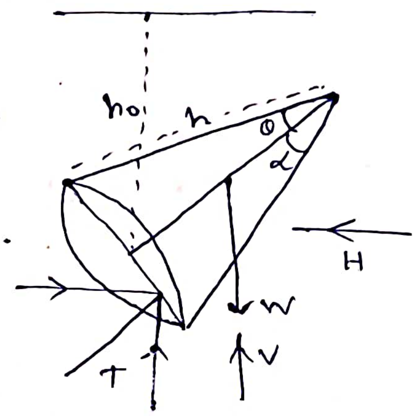
Thus the resultant pressure on the

curved surface is reduced to two forces, one horizontal passing through $H\left(\frac{5}{16}a, \frac{5}{16}b, \frac{3}{7}h\right)$ and the other vertical.

2015 A cone is totally immersed in water, the depth of the centre of its base being given. Prove that P, P', P'' being the resultant pressures on its convex surface, when the sine of the inclination of its axis to the horizon ~~to~~ are s, s', s'' respectively

$$P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0$$

Soln



Let h_0 be the depth of the centre of base below the effective surface. h be the hight of the cone & α be the semivertical angle.

Let a be the radius of the base.

$\therefore \frac{a}{h} = \tan\alpha$

$\Rightarrow a = h\tan\alpha$

Let 'T' be the thrust on the plane ~~below~~ base.

H be the resultant horizontal thrust

V be " " vertical "

on the curved surface & W be the wt of the displaced liquid.

Let θ be the angle of inclination of the axis of the cone of horizon

$$\therefore T = g\rho h_0 \pi a^2$$
$$= g\rho h_0 \pi h^2 \tan^2\alpha$$
$$= \lambda \text{ (say)}$$

$$H = T\cos\theta = \lambda\cos\theta$$

$$\therefore V = W - T\sin\theta = W - \lambda\sin\theta$$

$$\therefore R^2 = H^2 + V^2$$
$$= \lambda^2\cos^2\theta + (W-\lambda\sin\theta)^2$$
$$= \lambda^2\cos^2\theta + W^2 - 2\lambda W\sin\theta + \lambda^2\sin^2\theta$$
$$= \lambda^2 + W^2 - 2W\lambda\sin\theta$$
$$= \mu - \gamma\sin\theta$$

where $\mu = \lambda^2 + W^2$

$\gamma = 2W\lambda$

A/Q
$$P^2 = \mu - \gamma s$$
$$P'^2 = \mu - \gamma s'$$
$$P''^2 = \mu - \gamma s''$$

∴ L.H.S.

$(\mu - \gamma s)(s' - s'') + (\mu - \gamma s')(s'' - s) + (\mu - \gamma s'')(s - s')$

$= \mu s' - \mu s'' - \gamma s s' + \gamma s s'' + \mu s'' - \mu s - \gamma s' s'' + \gamma s s'$

$+ \mu s - \mu s' - \gamma s s'' + \gamma s' s''$

$= 0$ R.H.S. //

***Example 2.** *A hemispherical bowl is filled with water, and two vertical planes are drawn through its central radius, cutting off a semi-line of the surface ; if 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical* $= \tan^{-1}\left(\dfrac{\sin\alpha}{\alpha}\right)$

Sol. Let OAZ and OBZ be the two vertical planes through the central radius OZ, the angle AOB between the planes being 2α.

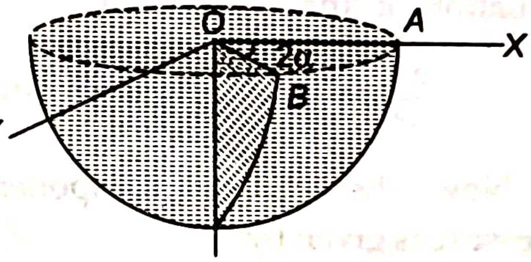


Fig. 146

Now the vertical thrust, Z on the curved surface is given by

Z = weight of the super-incumbent liquid

= weight of the liquid contained

$= \rho g . \dfrac{2}{3}\alpha a^3$ $\left(\text{as volume } OABZ = \dfrac{2}{3}\alpha a^3\right)$.

If Y be the horizontal thrust parallel to y-axis, then

Y = pressure on the face OAZ

$= \rho g . \left(\dfrac{1}{2}\pi a^2\right) . \dfrac{4a}{3\pi}$ $\left[\text{as depth of C.G. of a quadrant } = \dfrac{4a}{3\pi}\right]$

$= \dfrac{1}{3}a^3 \rho g.$

Similarly horizontal pressure Parallel to x-axis *i.e.,* the pressure on face OBZ is equal to $Y = \dfrac{1}{3}a^3 \rho g.$

These two equal forces, each equal to Y, act at right angles to the two faces which include an angle 2α between them; therefore the angle between these two Y's is $\pi - 2\alpha$.

Hence if H be the resultant horizontal thrust on the body, then

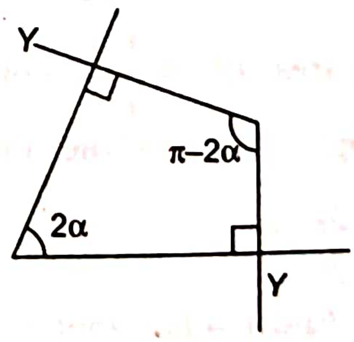


Fig. 147

$$H = \sqrt{[Y^2 + Y^2 + 2.Y.Y\cos(\pi - 2\alpha)]}$$

$$= Y\sqrt{(2 - 2\cos 2\alpha)} = \frac{1}{3}a^3\rho g\sqrt{(4\sin^2\alpha)} = \frac{2}{3}a^3\rho g \sin\alpha.$$

Now ϕ be the angle that the resultant thrust (resultant of H and Z now) makes with the vertical, then

$$\tan\phi = \frac{H}{Z} = \frac{\frac{2}{3}a^3\rho g\sin\alpha}{\frac{2}{3}a^3\alpha\rho g} = \frac{\sin\alpha}{\alpha}$$

or
$$\phi = \tan^{-1}\left(\frac{\sin\alpha}{\alpha}\right)$$

2018

***Example 3.** *A vessel full of water is in the form of an eighth part of an ellipsoid (axes a, b, c) boundary by the three principal planes. The axis c is vertical, and the atmospheric pressure is neglected. Prove that the resultant fluid pressure on the curved surface is a force of intensity.*

$$\frac{1}{8}\rho g[b^2c^4 + a^2c^4 + \frac{1}{4}\pi^2a^2b^2c^2]^{1/2}$$

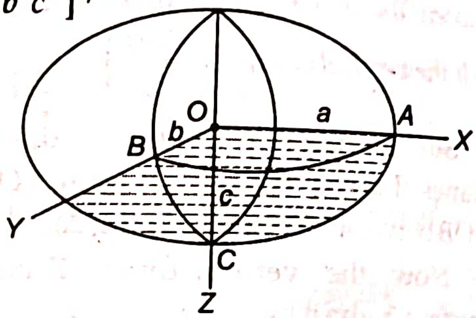


Fig. 148

Sol. Referred to O, the centre, as origin, the equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad ...(1)$$

Now the vertical component Z of the pressure is given by

Z = weight of the liquid contained

$$= \rho g\left(\frac{1}{8}\text{ Volume of the ellipsioid}\right) = \frac{1}{8}\cdot\frac{4}{3}\pi abc\rho g$$

$$= \frac{1}{8}\pi abc\rho g. \quad ...(2)$$

acting vertically downwards through the centre of gravity of the octant of the ellipsoid, *i.e.*, through, the point $\left(\frac{3}{8}a, \frac{3}{8}b, \frac{3}{8}c\right)$.

Again X = horizontal thrust along x-axis

= thrust on face OBC

$$= \rho g\left(\frac{1}{4}\pi bc\right)\cdot\frac{4c}{3\pi} = \frac{1}{3}\rho gbc^2$$

(as area $OBC = \frac{1}{4}$ area of an ellipse of semi-axes b and c) acting through the C.P. of the elliptic quadrant OBC. Proceeding as in Ex. 1, P .65, the coordinates of C.P. of OBC are

$$\left(0, \frac{3}{8}b, \frac{3}{16}\pi c\right).$$

And Y = horizontal thrust along y-axes

= thrust on face OAC

$$= \rho g\left(\frac{1}{4}\pi ac\right)\cdot\frac{4c}{3\pi} = \frac{1}{3}\rho gac^2$$

acting through the C.P. of face OAC *i.e.*, through $\left(\frac{3}{8}a, 0, \frac{3}{16}\pi c\right)$.

Now the vertical line through $\left(\frac{3}{8}a, \frac{3}{8}b, \frac{3}{8}c\right)$ line parallel to x-axis through $\left(0, \frac{3}{8}b, \frac{3}{16}\pi c\right)$ and a line parallel to y-axis through $\left(\frac{3}{8}a, 0, \frac{3}{16}\pi c\right)$ pass through a common point $\left(\frac{3}{8}a, \frac{3}{8}b, \frac{3}{16}\pi c\right)$.

Since X, Y, Z intersect at a common point, these can be reduced to a single force of magnitude

$$R = \sqrt{X^2 + Y^2 + Z^2}$$

$$= \sqrt{\left(\frac{1}{3}b^2c^4\rho^2g^2 + \frac{1}{9}a^2c^4\rho^2g^2 + \frac{1}{36}\pi^2a^2b^2c^2\rho^2g^2\right)}$$

$$= \frac{1}{3}\rho g\left[b^2c^4 + a^2c^4 + \frac{1}{4}\pi^2a^2b^2c^2\right]^{1/2}$$

Also line of action of the resultant of X, Y, Z acting through $\left(\frac{3}{8}a, \frac{3}{8}b, \frac{3}{16}\pi c\right)$ is given by

$$= \frac{x - \frac{3}{8}a}{X} = \frac{y - \frac{3}{8}b}{Y} = \frac{z - \frac{3}{16}c}{Z}$$

or $$= \frac{x - \frac{3}{8}a}{\frac{1}{3}\rho gbc^2} = \frac{y - \frac{3}{8}b}{\frac{1}{3}\rho gac^2} = \frac{z - \frac{3}{16}}{\frac{1}{6}\pi abc\rho g}$$

or $$= a\left(x - \frac{3}{8}a\right) = b\left(y - \frac{3}{8}b\right) = \frac{2c}{\pi}\left(z - \frac{3\pi c}{16}\right)$$ ●

2017

***Example 9.** *A closed cylinder, very nearly filled with liquid, rotates uniformly about a generating line, which is vertical; find the resultant pressure on its curved surface.*

Determine also the point of action of the pressure on its upper end.

Sol. Let the cylinder rotate about the vertical generator OZ with a constant angular velocity ω. Consider a point P at a distance u from OZ and at a depth z from O.

Then pressure at P is given by

$$dp = \rho[\omega^2 u\, du + g\, dz]$$

Integrating,

$$p = \rho[\frac{1}{2}\omega^2 u^2 + gz]. \qquad ...(1)$$

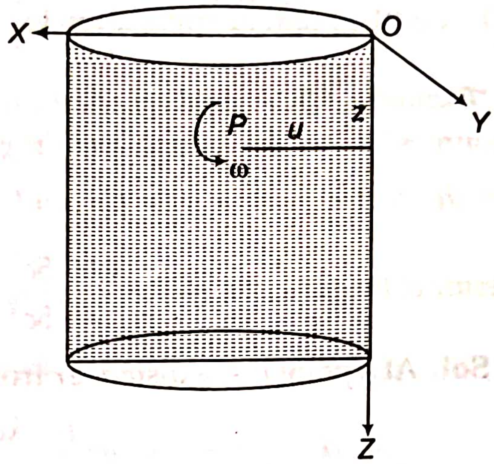


Fig. 154

The constant of integration vanishes as the cylinder is very nearly filled, so that at O when $u = 0, z = 0, p = 0$.

Let (u, ϕ, z) be any point on the surface of the cylinder. Then $u = 2a\cos\phi$ in is the equation of the cylinder.

Hence element of the surface $= u\,\delta\phi\,\delta z\; 2a\cos\phi\,\delta z$ and putting $u = 2a\cos\phi$ in (1) pressure at the point (u, ϕ, z)

$$= \rho\left[\frac{1}{2}\omega^2.4a^2\cos^2\phi + gz\right] = \rho(2\omega^2 a^2\cos^2\phi + gz).$$

$\therefore$ Total pressure on the curved surface

$$= 2a\rho\int_{\phi=-\pi/2}^{\pi/2}\int_{z=0}^{x=h}\cos\phi[2\omega^2 a^2\cos^2\phi + gz]d\phi\, dz$$

$$= 2a\rho\left[4a^2\omega^2 h\int_0^{\pi/2}\cos^3\phi\, d\phi + gh^3\int_0^{\pi/2}\cos\phi\, d\phi\right]$$

$$= 2a\rho\left[4a^2\omega^2.\frac{2}{3} + gh^2\right] = \frac{2}{3}a\rho h\left[8a^2\omega^2 + 3gh\right].$$

2nd part. Putting $z = 0$ in (1), the pressure at any point $(u, \phi, 0)$ on the upper end $= \frac{1}{2}\rho\omega^2 u^2$

Let the elementary area of the upper end enclosing $(u, \phi, 0)$ be $u\, d\theta\, \delta\phi$. Then the C.P. (the point of application of resultant pressure) of the upper face lies on the x-axis (diameter through O) at a distance $\bar{x}$ such that

$$\bar{x} = \frac{\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\phi} u\cos\phi\frac{1}{2}\rho\omega^2 u^2 u\, d\phi\, du}{\int_{-1/\pi}^{\pi/2}\int_0^{2a\cos\phi}\frac{1}{2}\rho\omega^2 u^2 u\, d\phi\, du}$$

$$\bar{x} = \frac{8a}{5}\frac{\int_{-\pi/2}^{\pi/2}\cos^6\phi\, d\phi}{\int_{\pi/2}^{\pi/2}\cos^4\phi\, d\phi} = \frac{8a}{5}\frac{\int_0^{\pi/2}\cos^6\phi\, d\phi}{\int_0^{\pi/2}\cos^4\phi\, d\phi}$$

$$= \frac{8a}{5} \frac{\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{7}{2}\right)}{2\Gamma 4} \div \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma 3} \frac{4a}{3}.$$

Hence the point of application is $\left(\frac{4a}{3}, 0, 0\right)$.