UNIT - 3

Floating bodies

Equilibrium of floating bodies:-

Archimedes principle:- When a body is partially or wholly immersed in a fluid at rest, the resultant thrust of the fluid on the solid is equal and opposite to the weight of the fluid displaced by the body and acts vertically upwards through the centre of gravity of the fluid displaced.

(06) (O)

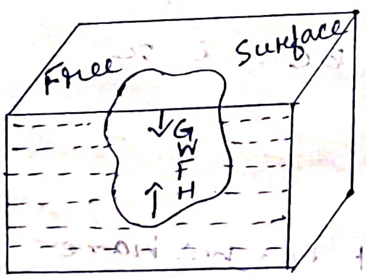
The weight of the liquid displaced by the body is known as the force of buoyancy and the centre of gravity of this displaced liquid is called the centre of buoyancy.

(06) (O)

If the weight of the body is greater than or equal to the weight of the liquid displaced, then the body will sink. If the weight of the body is less than the weight of the liquid displaced then the body will float. In any case, the C.G. of the displaced liquid is called the centre of buoyancy.

(09)M Conditions of equilibrium of a body:- Floating freely in a homogeneous liquid:-

(04, 03, 98) M



When a body is freely floating in a homogeneous fluid, the only two forces acting on it are:

1) Weight W of the body acting vertically downwards through G, the centre of gravity of the body.

2) The force of buoyancy F acting vertically upwards through H, the centre of buoyancy i.e the centre of gravity of the liquid displaced.

For equilibrium these two forces must balance each other. Thus the conditions of equilibrium for a body floating freely in a homogeneous liquid are:-

1) The weight of the displaced liquid must be equal to the weight of the body.

2) The centre of gravity of the body and the displaced liquid must be in the same vertical line.

Defns:-

1. plane of floatation:- (88, 89, 97, 99, 04, 09) M

A body floating in a homogeneous liquid partially immersed is intersected by the plane surface of the liquid. This plane is called the plane of floatation.

(88, 89, 94, 97, 04, 06) M

2. Centre of buoyancy:- The centre of gravity of the displaced liquid is called the centre of buoyancy. It is denoted by H.

(88, 89, 94, 99) M

3. Surface of Buoyancy:- If the body moves so that the volume of the liquid displaced remains unchanged, then the locus of H, the centre of buoyancy is the surface of buoyancy.

(88, 89, 94, 99) M

4. Surface of floatation:- If the body moves so that the volume of the liquid displaced remains unchanged, then the envelope of the planes of floatation is the surface of floatation.

(92, 97, 88, 95) M

5. The curves of floatation and curve of Boyancy:-

They are principal normal section of corresponding pts. on a surface of floatation and a surface of buoyancy.

Stability :- Let a body be floating partially immersed in a liquid.

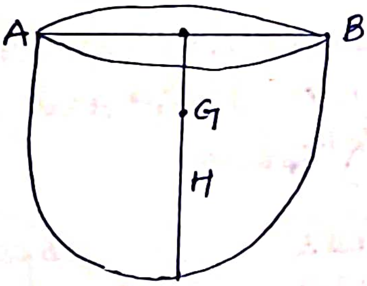
When slightly displaced ~~a~~ from its position of equilibrium ~~If~~ and the body tends to return to it's original position, then the equilibrium is said to be stable.

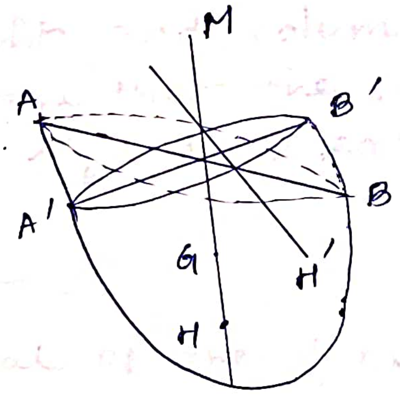
On the other hand if after being sightly displaced from the position of equilibrium; it tends to recede further from the original position, the equilibrium is unstable.

(08, 06, 03, 99) M

Meta - centre :-

Let a body float in equilibrium with AB as plane of floatation. In this position let G be the C.G. of the body and H the centre of buoyancy. Since the body is floating in equilibrium, GH is vertical.





Now, let the body be slightly rotated in such a way that the mass of the liquid displaced remains unchanged. In this position let H' be the centre of buoyancy. The pt. of intersection M of GH with the vertical through H' is called the meta centre.

2015
2019 **Example 16.** *A cone of given weight and volume floats with its vertex downwards, prove that the surface of the cone in contact with the liquid is least when its vertical angle* $2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Sol. For a cone with height h, semi vertical angle α and material density ρ be floating in a liquid of density σ.

Let x be the depth of the vertex O below the surface of floatation, then the weight of the cone is given by

$$W = \frac{1}{3}\pi r^2 h\rho g = \frac{\pi}{3}h^3 \tan^2\alpha\rho g = V\rho g,$$

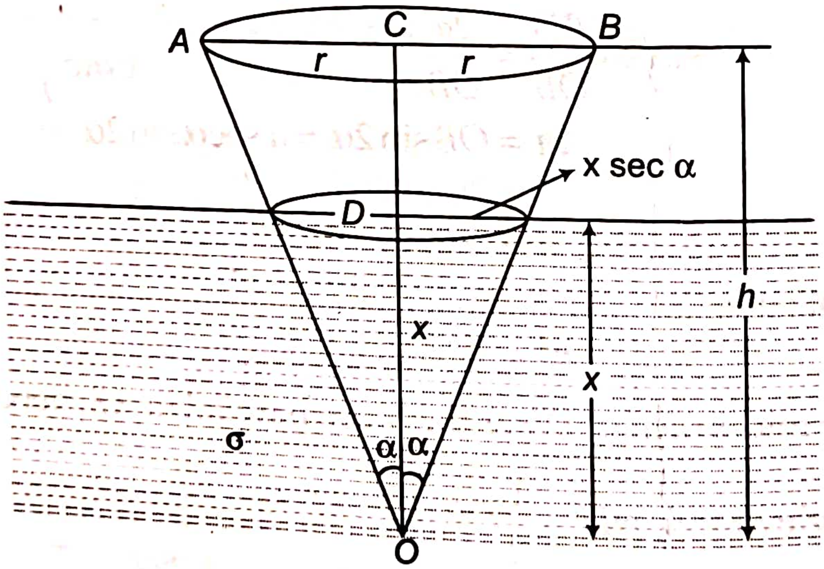


Fig. 221

Where volume of the cone is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} h^3 \tan^2\alpha \qquad \text{...(1)}$$

$$\therefore \quad \frac{W}{V} = \rho g \qquad \text{...(2)}$$

Since W and V are constant, hence ρ is constant.

As the cone is floating in equilibrium,

so weight of the cone = weight of the liquid displaced by the cone.

$$\Rightarrow \quad \frac{\pi}{3} h^3 \tan^2\alpha \rho g = \frac{\pi}{3} x^3 \tan^2\alpha \sigma g$$

$$\Rightarrow \quad h^3 \rho = x^3 \sigma \qquad \text{...(3)}$$

Let S be the area of the curved surface in contact with the liquid, then

$$S = \pi r l = \pi(x \tan\alpha)(x \sec\alpha) = \pi x^2 \tan\alpha.\sec\alpha$$

$$= \pi\left(\frac{h^3\rho}{\sigma}\right)^{3/2} \sec\alpha.\tan\alpha \qquad \text{[using...(3)]}$$

$$= \pi\left(\frac{\rho}{\sigma}\right)^{2/3} h^2 \sec\alpha \tan\alpha = \pi\left(\frac{\rho}{\sigma}\right)^{2\backslash 3}\left(\frac{3V}{\tan^2\alpha}\right)^{2\backslash 3} \sec\alpha.\tan\alpha \qquad \text{[using (1)]}$$

$$S = \left(\frac{9\pi\rho^2 V^2}{\sigma^2}\right)^{1/3} \tan^{-1/3}\alpha \sec\alpha. = k\frac{\sec\alpha}{(\tan\alpha)^{1/3}} \text{ where } k = \left(\frac{9\pi\rho^2 V^2}{\sigma^2}\right)^{1/3}$$

$$\frac{dS}{d\alpha} = k\left[\frac{\tan^{1/3}\alpha \sec\alpha \tan\alpha - \sec^3\alpha.\frac{1}{3}(\tan\alpha)^{-2/3}}{\tan^{2/3}\alpha}\right]$$

For maximum and minimum values of S,

$$\frac{dS}{d\alpha} = 0 \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad \text{Vertical angle} = 2\alpha = 2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

●

... ... greater in

2014
2016

***Example 2.** *A hollow hemispherical shell has a heavy particle fixed to its rim, and floats in water with the particle just above the surface and with the plane of the rim inclined at an angle of 45° to the surface. Show that the weight of the hemisphere is to the weight of the water which it would contain is* $(4\sqrt{2}-5):6\sqrt{2}$.

Sol. Let a be the radius of the hollow hemisphere ; then its weight W will act through G, where $OG = \frac{a}{2}$, G being on the symmetrical radius OC.

Let $PCAQ$ be the free surface, then from the question $\angle OAC = 45°$.
Since $OA = OC$, hence $\angle OCA = 45°$
$\therefore$ $\angle AOC = 90°$ *i.e.*, C is the vertex of the herispherical shell. Let W' be the weight of the particle at A.
$\therefore$ $W + W'$ = weight of the liquid displaced by the volume AMC.

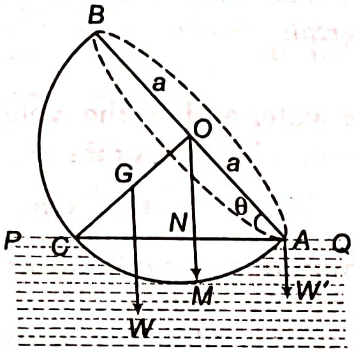


Fig. 242

Now volume of the segment AMC

$$= \pi MN^3\left[OM - \frac{MN}{3}\right] \qquad \left[\begin{aligned}&\text{as} \quad ON = a/\sqrt{2},\ OM = a.\\ &\therefore \quad MN = a - a/\sqrt{2}\\ &\qquad\quad = a\,(1 - 1/\sqrt{2})\end{aligned}\right]$$

$$= \pi a^2\left(1-\frac{1}{\sqrt{2}}\right)^2\left[a - \frac{1}{3}a\left(1-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi a^3}{6\sqrt{2}}(\sqrt{2}-1)^2\,[3\sqrt{2}-\sqrt{2}+1]$$

$$= \frac{\pi a^3}{6\sqrt{2}}(3-2\sqrt{2})(2\sqrt{2}+1)$$

$$= \frac{\pi a^4}{6\sqrt{2}}(4\sqrt{2}-5).$$

The upthrust = volume of the segment AMC × density of liquid × g

$$\therefore \quad W + W' = \frac{\pi a^3}{6\sqrt{2}}(4\sqrt{2}-5)\,\rho g \qquad \text{...(1)}$$

which acts vertically upwards through MN.

Now taking moment of all forces about O, we get $W.OG \sin 45° = W'\,OA \cos 45°$

$$W\,\frac{a}{2}\cdot\frac{1}{\sqrt{2}} = W'a\,\frac{1}{\sqrt{2}}$$

$$W = 2W' \qquad \text{...(2)}$$

From (1) and (2)

$$\frac{3}{2}W = \frac{\pi a^3}{6\sqrt{2}}(4\sqrt{2}-5)$$

Now taking moments about A where the particle is fixed, we get for equilibrium.

$$W \times \left(a + \frac{a}{2}\right) \cos 45° = \frac{\pi a^3}{6\sqrt{2}} (4\sqrt{2} - 5)\, \rho . a \cos 45°$$

or

$$\frac{W}{\frac{2}{3}\pi a^3 \rho} = \frac{4\sqrt{2} - 5}{6\sqrt{2}}.$$

i.e.,

$$\frac{\text{wt. of the hemispherical shell}}{\text{wt. of water it can contains}} = \frac{4\sqrt{2} - 5}{6\sqrt{2}} \Rightarrow \frac{W}{\frac{2}{3}\pi a^3 \rho g} = \frac{(4\sqrt{2} - 5)p}{6\sqrt{2}}$$

Theorems on Stability

2019
2017

§ 8.3 *To show that the, equilibrium condition is stable or unstable or neutral according as meta-centre is above or below the centre of gravity or coincides with the centre of gravity.*

[TMBU - 06 H, 09 H]

Proof. Let W be the weight of the floating body with centre of gravity at G, then W act vertically downwards through the centre of gravity G in any floating position of the body. Let H be the centre of buoyancy (*i.e.*, C.G. of the displaced liquid) in the position of equilibrium, through which weight W of the displaced liquid (*i.e.* upthrust of the displaced liquid) acts vertically upwards and GH is vertical.

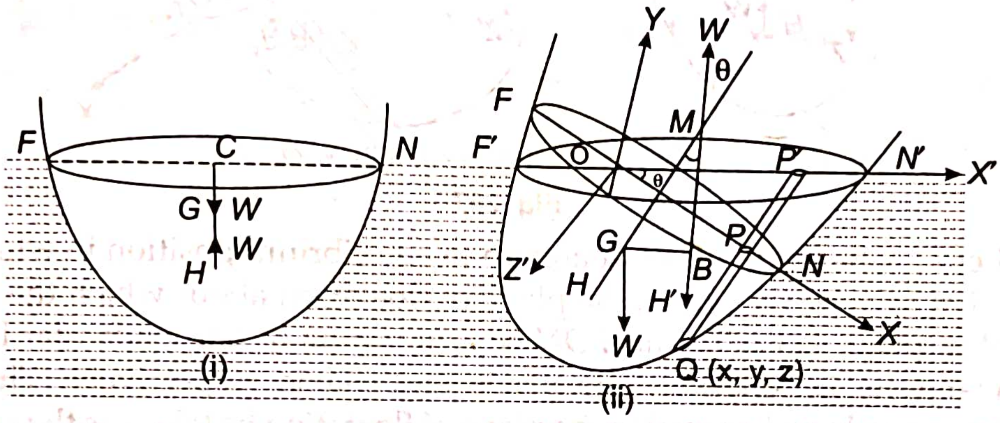


Fig. 266

Now let the body be slightly displaced through an angle θ and let H' be the displaced position of H. In the displaced position the upthrust W of the displaced liquid will act vertically upward through H'. Thus the weight W of the body through G and weight W of the displaced liquid through H' form a couple.

Drawing a vertical line through H', which intersects HG extended at M, the point M is called meta-centre of the body. Drawing a perpendicular GB from G on $H'M$ then in ΔMGB,

$$\therefore \qquad \sin\theta = \frac{GB}{GM} \quad \Rightarrow GB = GM \sin\theta$$

$$\text{Moment of the couple } (W, W) = W.GB$$
$$= W.GM \sin\theta$$
$$= W\,(HM - HG) \sin\theta.$$

If the moment of this couple is (+ve) i.e. the couple has a tendency to rotate the body in antilock-wise direction, then the body will tend to come back to its original position and if the moment of the couple is (–ve), then the couple has a tedency to rotate the body in clock-wise direction i.e. the body will tend to recede further from its displaced position.

$\therefore$ The equilibrium of the body will be stable or unstable, according as

$$(HM - HG)\sin\theta > \theta \text{ or } < 0$$

i.e. $\qquad HM - HG > 0$ or < 0 $\qquad$ [as $\sin\theta > 0$]

i.e. $\qquad HM >$ or $< HG$

i.e. equilibrium is stable if $HM > HG$ *i.e.* M is above G

and equilibrium is unstable if $HM < HG$ *i.e.* M is below G.

i.e. Equilibrium is stable of unstable according as M is above G or M is below G.

2018

§ 8.8 *To find the position of the meta-centre and hence deduce the meta-centric height.*

[CU 2001(H), 2003(H)]

Proof. Referring to the figure (266), let H and H' be the centres of buoyancy in original position and displaced position of the floating body respectively, then the coordinates of H and H' are given by

$$\left.\begin{aligned}\bar{x} &= \frac{1}{V}\iint xz\,dxdy\\ \bar{y} &= \frac{1}{V}\iint yz\,dxdy\\ \bar{z} &= \frac{1}{2V}\iint z^2\,dxdy\end{aligned}\right\}\quad \text{[as in § 8.5]} \qquad \ldots(1)$$

$$\left.\begin{aligned}\bar{x}' &= \frac{1}{V}\iint x\,(z+x\theta)\,dxdy\\ \bar{y}' &= \frac{1}{V}\iint y\,(z+x\theta)\,dxdy\\ \bar{z}' &= \frac{1}{2V}\iint (z^2 - x^2\theta^2)\,dxdy\end{aligned}\right\}\quad \text{[as in § 8.5]} \qquad \ldots(2)$$

From (1) and (2)

$$HH' = \bar{x}' - \bar{x}$$

$$= \frac{1}{V}\iint x\,(z+x\theta)\,dxdy - \frac{1}{V}\iint xz\,dxdy = \frac{1}{V}\iint x^2\theta\,dxdy$$

$$\Rightarrow \qquad HH' = \frac{\theta}{V}\iint x^2\,dxdy \qquad \ldots(3)$$

Let A be the area of the plane of floatation and k be the radius of gyration of the plane of floatation about the axis of rotation *i.e.* y-axis, then

$\iint x^2\,dxdy$ = moment of inertia of the plane of floatation FTN about the axis of rotation

$$= Ak^2 \qquad \ldots(4)$$

From equation (3) and (4),

$$HH' = \frac{\theta}{V} Ak^2$$

In $\Delta HH'M$, HH', $= HM \sin\theta = HM\theta$ [as θ is very small]

$$\Rightarrow \quad \frac{\theta}{V} Ak^2 = HM\theta$$

$$\Rightarrow \quad HM = \frac{Ak^2}{V} \quad \text{...(5)}$$

This gives the position of the meta-centre M *i.e.* the distance of the meta-centre from the centre of buoyancy H. Here HM depends upon the area of the plane of floatation in the original position and volume of the liquid displaced.

The metacentric height GM is given by

$$GM = HM - HG = \frac{Ak^2}{V} - HG. \quad \text{...(6)}$$

2015 **Example 2.** *If the floating solid be a cylinder, with its axis vertical, the ratio of whose specific gravity to that of the fluid is σ prove that the equilibrium will be stable, if the ratio of the radius of the base to the height be greater than*

$$[2\sigma(1-\sigma)]^{1/2}$$

(C.U. 02H)

Sol. Let a be the radius of the base and h the height of the cylinder of density ρ_1. If h' be the length of the axis immersed in the liquid of density ρ_2, then for equilibrium,

wt. of the cylinder = wt. of liquid displaced

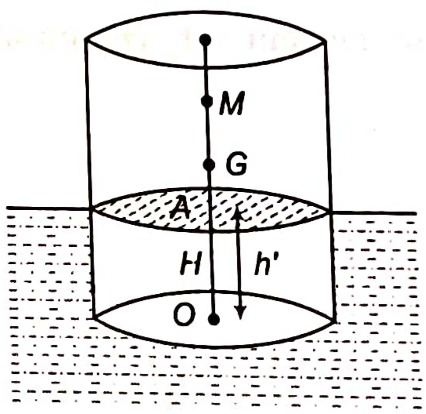


Fig. 269

i.e., $$\pi a^2 h\rho_1 = \pi a^2 h'\rho_2$$

where $$\frac{\rho_1}{\rho_2} = \sigma$$

or $$\frac{h'}{h} = \frac{\rho_1}{\rho_2} = \sigma \qquad ...(1)$$

The plane of floatation is a circle of radius a

$\therefore$ $$HM = \frac{Ak^2}{V} = \frac{\text{moment of inertia of the plane of flotation}}{\text{volume immersed}}$$

$$= \frac{\pi a^2 \cdot \frac{a^2}{4}}{\pi a^2 h'} = \frac{a^2}{4h'}$$

Also $$OH = \frac{h'}{2} \text{ and } OG = \frac{h}{2}$$

so that $$HG = OG - OH = \frac{1}{2}(h - h')$$

The equilibrium will be stable if

$$HM > HG$$

i.e., $$\frac{a^2}{4h'} > \frac{1}{2}(h-h') \text{ or } a^2 > 2h'(h-h').$$

or $$\frac{a^2}{h^2} > \frac{2h'}{h}\left(1 - \frac{h'}{h}\right) > 2\sigma(1-\sigma) \qquad \text{[from (1)]}$$

i.e., $$\frac{a}{h} > \sqrt{[2\sigma(1-\sigma)]}$$

This proves the result.

2017
2013 **Example 6.** *A solid cone is floating with its axis vertical and vertex downwards. Discuss its stability of equilibrium.* **[TMBU-13H]**

Sol. Let h be the height and α the sermi-vertical angle of the cone of density ρ. Let it float in the liquid of density σ with a length h' of its axis immersed.

Then for floating,

$$\text{wt. of the cone} = \text{wt. of the liquid displaced,}$$

i.e.,
$$\frac{1}{3}\pi h^3 \tan^2\alpha\cdot\rho = \frac{1}{3}\pi h'^3 \tan^2\alpha\cdot\sigma$$

or
$$\frac{h^3}{h'^3} = \frac{\sigma}{\rho} \qquad \text{...(1)}$$

The plane of flotation is a circle of radius $h'\tan\alpha$.

$\therefore$
$$HM = \frac{Ak^2}{V} = \frac{\pi h'^2 \tan^2\sigma\left(\frac{1}{2}h'^2\tan^3\alpha\right)}{\frac{1}{3}\pi h^3 \tan^2\alpha} = \frac{3}{4}h'\tan^2\alpha$$

Also $\quad OH = \frac{3}{4}h'$ and $OG = \frac{3}{4}h$

so that $\quad HG = OG - OH = \frac{3}{4}(h - h')$.

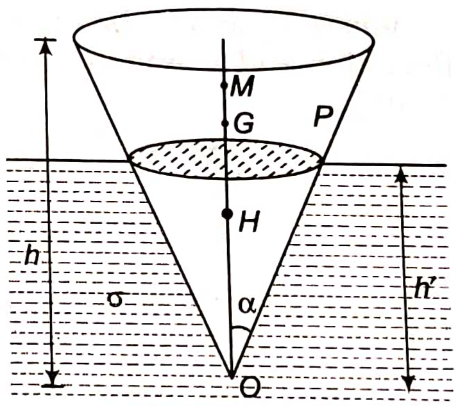


Fig. 272

The equilibrium will be stable if

$$HM > HG$$

or $$\frac{3}{4}h' \tan^2\alpha > \frac{3}{4}(h - h')$$

or $$h' \sec^2\alpha > h$$

or $$\frac{h}{h'} > \cos^2\alpha$$

or $$\frac{h'^3}{h^3} > \cos^6\alpha$$

or $$\frac{\rho}{\sigma} > \cos^6\sigma \qquad \text{[from (1)]}$$

2016 **Example 10.** *If a solid paraboloid, bounded by a plane perpendicular to its axis, float with its axis vertical and vertex immersed, the height of the meta-centre above the centre of gravity of the displaced liquid is equal to half the latus rectum.*

Sol. Let the equation of the paraboloid of revolution be

$$\left.\begin{aligned} x^2+y^2 &= 4az \\ u^2 &= 4az, \end{aligned}\right\} \quad ...(1)$$

i.e.,

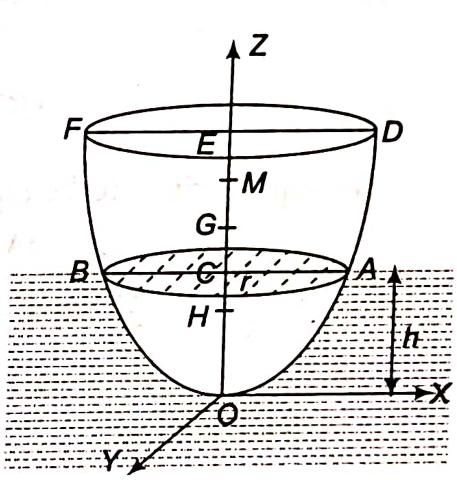


Fig. 275

where $4a$ is the latus rectum.

Let a length h of the axis of the paraboloid be immersed in the liquid.

Putting $z = h$ in (1), the radius of the plane of flotation which is a circle is given by

$$r^2 = 4ah \quad ....(2)$$

Ak^2 = moment of inertia of the plane of flotation about a principal axis, *i.e.*, diameter ACB

$$= \pi r^2 \cdot \frac{r^2}{4} = 4\pi a^2 h^2 \quad \text{[from (2)]}$$

Also V = volume immersed

$$= \int_0^h \pi u^2\, dz = \int_0^h \pi 4az\, dz = 2\pi a h^2$$

So, HM = height of meta-centre above the C.G, of the displaced liquid

$$= \frac{Ak^2}{V} = \frac{4\pi a^2 h^2}{2\pi a h^2} = 2a.$$

= half the latus rectum.

2016 **Example 17.** *A solid body consists of a right cone joined to a hemisphere on the same base and floats with the spherical portion partly immersed ; prove that the greatest height of the cone consistent with stability is $\sqrt{3}$ times the radius of the base.*

Sol. Let h be the height and a the radius of the base of the cone. Also let G_1 and G_2 be the centres of gravity of the hemisphere and the cone ; then

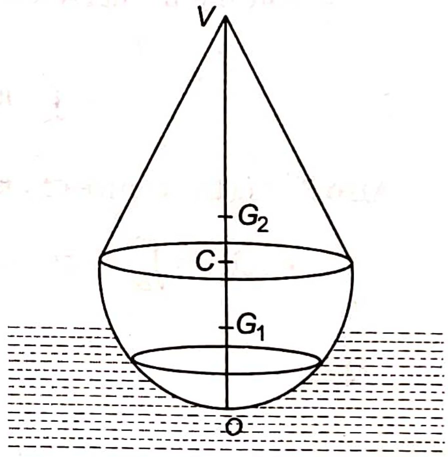


Fig. 280

$$OG_1 = OC - CG_1 = a - \frac{3}{8}a = \frac{5}{8}a,$$

$$OG_2 = OC + CG_2 = a + \frac{1}{4}h.$$

Hence if G be the joint centre of gravity, then

$$OG = \frac{\frac{2}{3}\pi a^3 \cdot \frac{5}{8}a + \frac{1}{3}\pi a^2 h \left(a + \frac{1}{4}h\right)}{\frac{2}{3}\pi a^3 + \frac{1}{3}\pi a^2 h}$$

$$= \frac{\frac{5}{4}a^2 + h\left(a + \frac{1}{4}h\right)}{2a + h}$$

But since the meta-centre is the centre of curvature of the surface of buoyancy, the centre C is the meta-centre of the body here.

For stability $OG \leq OC$

or $$\frac{5}{4}a^2 + h\left(a + \frac{1}{4}h\right) \leq a\,(2a + h)$$

or $$\frac{1}{4}h^2 < \frac{3}{4}a^2 \quad \text{or} \quad h \leq \sqrt{3}a.$$

This proves the result.