Section 3.4 Subsequences and the Bolzano-Weierstrass Theorem

In this section we will introduce the notion of a subsequence of a sequence of real numbers. Informally, a subsequence of a sequence is a selection of terms from the given sequence such that the selected terms form a new sequence. Usually the selection is made for a definite purpose. For example, subsequences are often useful in establishing the convergence or the divergence of the sequence. We will also prove the important existence theorem known as the Bolzano-Weierstrass Theorem, which will be used to establish a number of significant results.

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$\left(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots\right)$$

is called a **subsequence** of X.

For example, if $X := \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\right)$, then the selection of even indexed terms produces the subsequence

$$X' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2k}, \cdots\right),$$

where $n_1 = 2, n_2 = 4, \cdots, n_k = 2k, \cdots$. Other subsequences of $X = (1/n)$ are the following:

$$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2k-1}, \cdots\right), \quad \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \cdots, \frac{1}{(2k)!}, \cdots\right).$$

The following sequences are *not* subsequences of $X = (1/n)$:

$$\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \cdots\right), \quad \left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \cdots\right).$$

Theorem: If a sequence $X = \langle x_n \rangle$ of real numbers converges to a real number x, then any subsequence $X' = \langle x_{n_k} \rangle$ of X also converges to x.

Pf. Let the sequence $X = \langle x_n \rangle$ converges to x.
Let $\varepsilon > 0$ be given.
Then there exists a natural number $K(\varepsilon)$ such that $|x_n - x| < \varepsilon$, $\forall\ n \geq K(\varepsilon)$ — (1)

Let $X' = \langle x_{n_k} \rangle$ be any arbitrary subsequence of $X = \langle x_n \rangle$.
Then $n_1 < n_2 < n_3 < \dots < n_k < \dots$
is an increasing sequence of ~~real~~ natural numbers.
We can show that $n_k \geq k$, for all $k \in \mathbb{N}$.
For $k = 1$, $n_1 \geq 1$ [$\because n_1$ is an natural no.)
So the result is true for $k = 1$.
Let the result be true for $k = m$ i.e. $n_m \geq m$

Now, $n_{m+1} > n_m \geq m$

$\Rightarrow n_{m+1} > m \Rightarrow n_{m+1} \geq m+1$.

$\therefore$ the result is true for $k = m+1$. Hence by principle of induction the result is true for all k. i.e. $n_k \geq k$ for all $k \in \mathbb{N}$.
Thus if $k \geq K(\varepsilon)$, then $n_k \geq k \geq K(\varepsilon)$
and so $|x_{n_k} - x| < \varepsilon$ (from (1))
$\Rightarrow$ the subsequence $\langle x_{n_k} \rangle$ converges to x

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Ex. Show that $\lim(b^n) = 0$ if $0 < b < 1$.

Soln: Let $x_n = b^n$.

$\because$ $0 < b < 1$, so $b^{n+1} < b^n$, $\forall n \in \mathbb{N}$.

$\Rightarrow x_{n+1} < x_n$, $\forall n \in \mathbb{N}$.

$\Rightarrow \langle x_n \rangle$ is a decreasing sequence.

Also, $0 < b < 1 \Rightarrow 0 < b^n < 1$

$\Rightarrow 0 < x_n < 1$, $\forall n \in \mathbb{N}$.

Thus $\langle x_n \rangle$ is a bounded decreasing sequence, so by monotone convergence theorem $\langle x_n \rangle$ converges.

Let $x = \lim(x_n)$.

Since $\langle x_{2n} \rangle$ is a subsequence of $\langle x_n \rangle$

so $x = \lim(x_{2n})$.

But $x_{2n} = b^{2n} = (b^n)^2 = x_n^2$

$\therefore$ $x = \lim(x_{2n}) = \lim(x_n^2)$

$= \lim(x_n \cdot x_n) = \lim(x_n)\lim(x_n)$

$\Rightarrow x = x \cdot x$

$\Rightarrow x = x^2$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or 1.

Since the sequence $\langle x_n \rangle$ is decreasing and bounded above by $b < 1$, so x cannot be equal to 1. Hence x must be equal to 0.

$\therefore$ $x = 0 \Rightarrow \lim(x_n) = 0$

$\Rightarrow \lim(b^n) = 0$, $0 < b < 1$ #

Ex. Show that $\lim (c^{1/n}) = 1$ for $c > 1$.

Soln. Let $x_n = c^{1/n}$

As $c > 1$, so $x_n > 1$ and $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$

Thus $\langle x_n \rangle$ is bounded below and decreasing sequence. So by monotone convergence theorem the sequence $\langle x_n \rangle$ converges. Let $x = \lim (x_n)$

Since $\langle x_{2n} \rangle$ is a subsequence of $\langle x_n \rangle$ so

$\lim (x_{2n}) = x$.

Now, $x_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = (x_n)^{1/2}$

$\Rightarrow \lim (x_{2n}) = \lim (x_n)^{1/2}$

$\Rightarrow x = x^{1/2}$

$\Rightarrow x^2 = x$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or 1.

Since $x_n > 1$ for all $n \in \mathbb{N}$, so x cannot be equal to 0. Hence x must be equal to 1.

$\therefore x = 1 \Rightarrow \lim (x_n) = 1$

$\Rightarrow \lim (c^{1/n}) = 1$, for $c > 1$.

Theorem:- Let $X = \langle x_n \rangle$ be a sequence of real numbers. Then the following are equivalent.

(i) The sequence $X = \langle x_n \rangle$ does not converge to $x \in \mathbb{R}$

(ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$

(iii) There exists an $\varepsilon_0 > 0$ and a subsequence $X' = \langle x_{n_k} \rangle$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$

Pf. (i) $\Rightarrow$ (ii). Let $X = \langle x_n \rangle$ does not converge. Then for some $\varepsilon_0 > 0$, it is impossible to find a natural number K such that for all $n \geq K$ the terms x_n satisfy $|x_n - x| < \varepsilon_0$. That is for each $K \in \mathbb{N}$ it is not true that for all $n \geq K$, the inequality $|x_n - x| < \varepsilon_0$ holds i.e. for each $K \in \mathbb{N}$, there exists a natural number $n_K \geq K$ such that $|x_{n_K} - x| \geq \varepsilon_0$

(ii) $\Rightarrow$ (iii)

Let there exist an $\varepsilon_0 > 0$ such that for each $K \in \mathbb{N}$, there exists $n_K \in \mathbb{N}$ such that $n_K \geq K$ and $|x_{n_K} - x| \geq \varepsilon_0$.

~~Let $n_1 \in \mathbb{N}$~~ Thus for the natural number 1, ~~$n_1 \geq 1$~~. Let n_1 be the natural number such that $n_1 \geq 1$ and $|x_{n_1} - x| \geq \varepsilon_0$.

Let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$.

$\Rightarrow n_2 > n_1 \geq 1 \Rightarrow n_2 > 1 \Rightarrow n_2 \geq 2$.

Thus for the natural number 2, n_2 is the natural number such that $n_2 > n_1$ and $|x_{n_2} - x| \geq \varepsilon_0$.

Continuing in this way we obtain a subsequence $X' = \langle x_{n_K} \rangle$ of X such that

$|x_{n_K} - x| \geq \varepsilon_0$ for all $K \in \mathbb{N}$.

(iii) $\Rightarrow$ (i). Suppose $X = \langle x_n \rangle$ has a subsequence $X' = \langle x_{n_K} \rangle$ satisfying the condition in (iii).

$\Rightarrow$ the subsequence $X' = \langle x_{n_K} \rangle$ does not converge.

Hence the sequence $X = \langle x_n \rangle$ can not converge because if $X = \langle x_n \rangle$ converge then its sub sequence $X' = \langle x_{n_k} \rangle$ will also converge, but it is not possible.

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3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.*

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \cdots, x_{m_k}, \cdots$. Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots.$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \cdots, x_{m_r}$. Let $s_1 := m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (x_{s_k}) of X.

Q.E.D.

The Bolzano - Weierstrass Theorem.

A bounded sequence of real numbers has a convergent subsequence.

Pf: Let $X = \langle x_n \rangle$ be a bounded sequence of real numbers. So $\exists$ $M > 0$ s.t.

$$|x_n| < M \quad , \text{for all } n \in \mathbb{N}.$$

But monotone subsequence theorem the sequence $X = \langle x_n \rangle$ has a subsequence that is monotone. say $X' = \langle x_{n_k} \rangle$

Also, $|x_{n_k}| < M$, for all $n_k \in \mathbb{N}$.

Thus $X' = \langle x_{n_k} \rangle$ is a bounded monotone sequence. So by monotone convergence theorem the sequence $X' = \langle x_{n_k} \rangle$ is convergent.

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