Linear Equations with constant coefficients:

Some useful results: Let D stand for $\frac{d}{dx}$; D^2 for $\frac{d^2}{dx^2}$; and so on. The symbols D, D^2, etc are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example, $D^4 x^4$ shows that we must differentiate x^4 four times. Thus $D^4 x^4 = 24$. The following results are valid for such operators:

(1) $D^m + D^n = D^n + D^m$

(2) $D^m D^n = D^{m+n}$

(3) $D(u+v) = Du + Dv$, where u and v are functions of x.

(4) $(D-\alpha)(D-\beta) = (D-\beta)(\beta-\alpha)$, where α and β are constants.

D^{-1} is equivalent to an integration. For example $D^{-1}x^2 = \int x^2\,dx = \frac{x^3}{3}$. But it is important to note that the main object of D^{-1} is to find an integral but not the complete integral. Consequently the arbitrary constant which arises in integration must be omitted. The index of D^{-1}, say $(D^{-1})^4$ is denoted by D^{-4}. The negative index of D indicates the number of times the operation of integration is to be carried out. For example

$$D^{-2}x^2 = \int\left[\int x^2\,dx\right]dx = \int \frac{x^3}{3}\,dx = \frac{x^4}{12}$$

It is usual to write $1/D^m$ for D^{-m}. It is to be remembered that $DD^{-1} = 1$ and the symbol D with negative indices also satisfy the above-mentioned four results.

Furthermore we write

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = (D^2 + a_1 D + a_2) y = f(D) y,$$

where $f(D)$ is the operator now. If $f_1(D)$ and $f_2(D)$ be two operators, then $f_1(D) f_2(D)$ is also an operator such that

$$f_1(D) f_2(D) = f_2(D) f_1(D)$$

Again, if u be a function of x and K be a constant then

$$f(D)(Ku) = K f(D) u$$

Linear differential equations with constant coefficients:

A differential equation of the form

$$\frac{d^m y}{dx^m} + a_1 \frac{d^{m-1}y}{dx^{m-1}} + a_2 \frac{d^{m-2}y}{dx^{m-2}} + \cdots + a_m y = X \quad \cdots (1)$$

where X is a function of x only and $a_1, a_2, \ldots, a_m$ are constants, is called a linear differential equation of mth order.

Using the symbols $D, D^2, \ldots D^m$, (1) becomes

$$D^m y + a_1 D^{m-1} y + a_2 D^{m-2} y + \cdots + a_m y = X$$

$$\Rightarrow (D^m + a_1 D^{m-1} + a_2 D^{m-2} + \cdots + a_m) y = X \quad \cdots (2)$$

$$\Rightarrow f(D) y = X \quad \cdots (3)$$

where $f(D) = D^m + a_1 D^{m-1} + a_2 D^{m-2} + \cdots + a_m \quad \cdots (4)$

Consider the differential equation

$$f(D) y = 0 \quad \cdots (5)$$

obtained on replacing the right hand side of (3) by zero. We will now show that if $y_1, y_2, \ldots, y_m$ are m linearly independent solutions of (5) then, $c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$ is also a solution of (5); $c_1, c_2, \ldots, c_m$ being arbitrary constants.

Since $y_1, y_2, \dots, y_n$ are solutions of (5) we have

$$f(D)y_1 = 0, \quad f(D)y_2 = 0, \quad \dots, \quad f(D)y_n = 0 \quad \dots (6)$$

If $c_1, c_2, \dots, c_n$ are n arbitrary constants,

we get $\quad f(D)(c_1y_1 + c_2y_2 + \dots + c_ny_n)$

$$= f(D)(c_1y_1) + f(D)(c_2y_2) + \dots + f(D)(c_ny_n)$$

$$= c_1 f(D)y_1 + c_2 f(D)y_2 + \dots + c_n f(D)y_n$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_n \cdot 0 \quad [\text{using } (6)]$$

$$= 0$$

This proves the statement made above.

Since the general solution of a differential equation of the nth order contains n arbitrary constants, we conclude that

$$c_1y_1 + c_2y_2 + \dots + c_ny_n = u \text{ (say) is the}$$

general solution of (5).

Thus $f(D)u = 0 \quad \dots (7)$

Again, let v be any particular solution of (3) and hence we have $\quad f(D)v = X \quad \dots (8)$

Now, we have, $\quad f(D)(u+v) = f(D)u + f(D)v$

$$= 0 + X, \text{ using (7) and (8)}$$

This shows that $(u+v)$ i.e. $c_1y_1 + c_2y_2 + \dots + c_ny_n + v$ is the general solution of (3) i.e. (1)

Thus, the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$, where C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constants.