* Every equation of n dimensions has n roots and no more.

Relations between the roots and Co-efficients of equations.

Consider the equation

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \cdots + P_{n-1} x + P_n = 0. \quad \text{(1)}$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n roots of the equation (1). Then we have

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \cdots + P_{n-1} x + P_n$$
$$\equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) \quad (*)$$

When the factors of the second member of this identity are multiplied together, the highest power of x in the product is x^n; the Co-efficient of x^{n-1} is the sum of the n quantities $-\alpha_1, -\alpha_2 \ldots -\alpha_n$; the Coefficient of x^{n-2} is the is the sum of the products of these quantities taken two by two; the Coefficients of x^{n-3} is the sum of their products taken three by three and so on, the last term being the product of all the roots with their sign changed. ~~Eque~~

$\therefore$ Equating the Coefficients of x on each side of the identity $(*)$, we have the following series of equations:—

$$P_1 = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

$$P_2 = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \cdots + \alpha_{n-1}\alpha_n)$$

$$P_3 = -(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \cdots + \alpha_{n-2}\alpha_{n-1}\alpha_n)$$

$$\vdots$$

$$P_n = (-1)^n \alpha_1\alpha_2 \cdots \alpha_n.$$

Consider the cubic equation

$$x^3 + P_1 x^2 + P_2 x + P_3 = 0.$$

Let α, β, γ be the roots of this equation

Then $P_1 = -(\alpha + \beta + \gamma) \rightarrow (2)$

$P_2 = \alpha\beta + \alpha\gamma + \beta\gamma \rightarrow (3)$

$P_3 = -\alpha\beta\gamma \rightarrow (4)$

Now, $\alpha^2 \times (2) + \alpha \times (3) + 4 \Rightarrow$

$$P_1\alpha^2 + P_2\alpha + P_3 = -\alpha^3 - \alpha^2\beta - \alpha^2\gamma + \alpha^2\beta + \alpha^2\gamma + \alpha\beta\gamma - \alpha\beta\gamma$$

$$\Rightarrow P_1\alpha^2 + P_2\alpha + P_3 = -\alpha^3$$

$$\Rightarrow \alpha^3 + P_1\alpha^2 + P_2\alpha + P_3 = 0,$$

which is the given cubic equation with α in the place of x.

Ex. Solve the equation $x^3 - 5x^2 - 16x + 80 = 0$ the sum of two of its roots being equal to zero.

Soln: Let the roots be α, β, γ. Then

$$\alpha + \beta + \gamma = 5 \quad \text{--- (1)}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -16 \quad \text{--- (2)}$$

$$\alpha\beta\gamma = -80 \quad \text{--- (3)}$$

Let $\beta + \gamma = 0$. Then $\beta = -\gamma$

Then equation (1) $\Rightarrow \alpha + 0 = 5 \Rightarrow \alpha = 5$

equation (3) $\Rightarrow \alpha\beta\gamma = -80$

$\Rightarrow 5.(-\gamma).\gamma = -80$

$\Rightarrow \gamma^2 = 16 \Rightarrow \gamma = 4, -4$

If $\gamma = 4$, then $\beta = -4$

If $\gamma = -4$, then $\beta = 4$

$\therefore$ the roots are $5, 4, -4$

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Ex. Solve the equation $x^3 - 3x^2 + 4 = 0$, two of its roots being equal.

Soln: Let α, α, β be the roots of the equation. Then

$$\alpha + \alpha + \beta = 3 \Rightarrow 2\alpha + \beta = 3 \quad \text{--- (1)}$$

$$\alpha^2 + \alpha\beta + \beta\alpha = 0 \Rightarrow \alpha^2 + 2\alpha\beta = 0 \quad \text{--- (2)}$$

From equation (1) $\beta = 3 - 2\alpha$

equation (2) $\Rightarrow \alpha^2 + 2\alpha(3 - 2\alpha) = 0$

$\Rightarrow \alpha^2 + 6\alpha - 4\alpha^2 = 0$

$\Rightarrow 6\alpha - 3\alpha^2 = 0$

$\Rightarrow 3\alpha(2-\alpha) = 0$

$\therefore \alpha = 0$ or $\alpha = 2$.

But $\alpha = 0$ is not a root of the equation.

So $\alpha = 2$.

$\Rightarrow \beta = 3 - 2\alpha = 3 - 2.2 = -1$.

$\therefore$ The roots are $2, 2, -1$ //

Ex. The equation $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ has two pairs of equal roots, find them.

Soln. Let the roots be $\alpha, \alpha, \beta, \beta$. Then

$\alpha + \alpha + \beta + \beta = -4 \Rightarrow 2\alpha + 2\beta = -4$

$\Rightarrow \alpha + \beta = -2$ — (1)

$\alpha^2 + \alpha\beta + \alpha\beta + \alpha\beta + \alpha\beta + \beta^2 = -2$

$\Rightarrow \alpha^2 + \beta^2 + 4\alpha\beta = -2$. — (2)

From equation (1), $\alpha + \beta = -2 \Rightarrow \beta = -2 - \alpha$

$\therefore$ equation (2) $\Rightarrow$

$\alpha^2 + (-2-\alpha)^2 + 4\alpha(-2-\alpha) = -2$

$\Rightarrow \alpha^2 + 4 + \alpha^2 + 4\alpha - 8\alpha - 4\alpha^2 = -2$

$\Rightarrow -2\alpha^2 - 4\alpha + 6 = 0$

$\Rightarrow \alpha^2 + 2\alpha - 3 = 0$

$\Rightarrow (\alpha + 3)(\alpha - 1) = 0$

$\Rightarrow \alpha = -3, 1$

If $\alpha = -3$ then $\beta = -2-\alpha = -2-(-3) = 1$

if $\alpha = 1$, then $\beta = -2-\alpha = -2-1 = -3$.

$\therefore$ the roots are $1, 1, -3, -3$

HW

Ex. Solve the equation

$$x^3 - 9x^2 + 14x + 24 = 0$$

two of whose roots are in the ratio of 3 to 2.

Ex. Solve the equation $x^3 - 9x^2 + 23x - 16 = 0$ whose roots are in arithmetic progression.

Soln: Let the roots be $\alpha - d, \alpha, \alpha + d$. Then

$$(\alpha - d) + \alpha + (\alpha + d) = 9 \Rightarrow 3\alpha = 9 \Rightarrow \alpha = 3$$

$$(\alpha - d)\alpha(\alpha + d) = 16$$

$$\Rightarrow \cancel{(\alpha^2 - d^2).\alpha = 16}$$

$$\Rightarrow \cancel{(3^2 - d^2).3 = 16}$$

$$\Rightarrow \cancel{9 - d^2} = \frac{16}{3} \Rightarrow d^2 = 9 - \frac{16}{3} = \frac{27-16}{3}$$

$$\Rightarrow d^2 = \frac{11}{3}$$

$(\alpha - d).\alpha + d(\alpha+d) + (\alpha^2 - d^2) = 23$

$$(\alpha - d)\alpha + \alpha(\alpha + d) + (\alpha + d)(\alpha - d) = 23$$

$$\Rightarrow \alpha(\alpha - d + \alpha + d) + (\alpha^2 - d^2) = 23$$

$$\Rightarrow 2\alpha^2 + \alpha^2 - d^2 = 23$$

$$\Rightarrow 3\alpha^2 - d^2 = 23$$

$$\Rightarrow d^2 = 3\alpha^2 - 23 = 3.3^2 - 23 = 4$$

$$\Rightarrow d = \pm 2.$$

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For $d = 2$, $\therefore \alpha - d = 3 - 2 = 1$, $\alpha = 3$, $\alpha + d = 3 + 2 = 5$

For $d = -2$, $\alpha - d = 3 - (-2) = 5$; $\alpha = 3$

$\alpha + d = 3 - 2 = 1$

$\therefore$ the roots are $1, 3, 5$

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(HW)

Ex. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression.

Soln. Let the roots be $\frac{\alpha}{r}, \alpha, \alpha r$.

~~Then~~ the given equation is

$$27x^3 + 42x^2 - 28x - 8 = 0$$

$$\Rightarrow x^3 + \frac{42}{27}x^2 - \frac{28}{27}x - \frac{8}{27} = 0$$

$$\therefore \frac{\alpha}{r} \cdot \alpha \cdot \alpha r = \frac{8}{27} \Rightarrow \alpha^3 = \left(\frac{2}{3}\right)^3$$

$$\Rightarrow \alpha = \frac{2}{3}$$

Also, $\frac{\alpha}{r} + \alpha + \alpha r = \cancel{\frac{42}{27}} - \frac{42}{27}$

$$\Rightarrow \alpha\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$$

$$\Rightarrow \frac{1 + r + r^2}{r} = -\frac{\overset{21}{\cancel{42}}}{\underset{9}{\cancel{27}}} \times \frac{3}{\cancel{2}} = -\frac{7}{3}$$

$$\Rightarrow 3 + 3r + 3r^2 = -7r$$

$$\Rightarrow 3r^2 + 10r + 3 = 0$$

$$\therefore r = \frac{-10 \pm \sqrt{100 - 4.3.3}}{2.3} = \frac{-10 \pm \sqrt{64}}{6}$$

$$= \frac{-10 \pm 8}{6} = \frac{-10+8}{6}, \frac{-10-8}{6}$$

$$= -\frac{1}{3}, -3.$$

for $x = -\frac{1}{3}$, roots are

$\frac{\alpha}{x} = \frac{2}{3} \times (-3) = -2$

$\alpha = \frac{2}{3}$ and $\alpha x = \frac{2}{3} \times (-\frac{1}{3}) = -\frac{2}{9}$

for $x = -3$, $\frac{\alpha}{x} = \frac{2}{3} \cdot (-\frac{1}{3}) = -\frac{2}{9}$

$\alpha = \frac{2}{3}$ and $\alpha x = \frac{2}{3} \times (-3) = -2$

$\therefore$ roots are $-2, \frac{2}{3}, -\frac{2}{9}$ //

(HW)

Ex. Solve the equation $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$ whose roots are in geometric progression.

Sol^n Hint. Let the roots be $\frac{\alpha}{r^3}, \frac{\alpha}{r}, \alpha r, \alpha r^2$

(HW)

Ex. Solve the equation, $6x^3 - 11x^2 + 6x - 1 = 0$ whose roots are in harmonic progression.

Sol^n: Hint.! The given equation is

$$6x^3 - 11x^2 + 6x - 1 = 0$$

$$\Rightarrow x^3 - \frac{11}{6}x^2 + x - \frac{1}{6} = 0$$

Let α, β, γ be the roots. Since α, β, γ are in harmonic progression so $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in arithmetic progression.

$$\therefore \frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta}$$

$$\Rightarrow \frac{\gamma + \alpha}{\alpha\beta} = \frac{2}{\beta}$$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\beta$$

Depression of an equation when a relation exists between two of its roots:

Consider the equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0 \quad \text{--- (1)}$$

Let α and β be two roots of equn (1) such that $\beta = \phi(\alpha)$.

Substituting $\phi(x)$ for x in (1), we get

$$f(\phi(x)) \equiv a_0 (\phi(x))^n + a_1 (\phi(x))^{n-1} + \cdots + a_n$$

$$\Rightarrow F(x) \equiv a_0(\phi(x))^n + a_1 (\phi(x))^{n-1} + \cdots + a_n$$

putting α for x we get

$$F(\alpha) \equiv a_0 (\phi(\alpha))^n + a_1 (\phi(\alpha))^{n-1} + \cdots + a_n$$

$$\equiv f(\phi(\alpha))$$

$$\equiv f(\beta) = 0$$

$\therefore$ α is also a root of $F(x) = 0$.

Thus α is a common root of $f(x)$ and $F(x)$.

This α can be determined and from if $\beta = \phi(\alpha)$ can be determined.

Ex. The equation $x^3-5x^2-4x+20=0$ has two roots whose difference $=3$; find them.

Soln: Let α, β, γ be the roots of the equation. Let $\beta-\alpha=3$. Then $\beta=3+\alpha$ ...(i)
Substitute $x+3$ for x in the equation
$x^3-5x^2-4x+20=0$, we get ——(1)

$$(x+3)^3-5(x+3)^2-4(x+3)+20=0$$

$$\Rightarrow x^3+3x^2.3+3.x.3^2+3^3-5(x^2+6x+9)$$
$$-4x-12+20=0$$

$$\Rightarrow x^3+9x^2+27x+27-5x^2-30x-45$$
$$-4x+8=0$$

$$\Rightarrow x^3+4x^2-7x-10=0 \quad \text{——(ii)}$$

Since $x=2$ satisfies both equations (1) and (ii) so so common measure of (1) and (ii) is $x-2$.

$$\therefore \alpha=2.$$
$$\Rightarrow \beta=3+\alpha=3+2=5.$$

Also $\alpha+\beta+\gamma=5$
$$\Rightarrow 2+5+\gamma=5$$
$$\Rightarrow \gamma=-2$$
$\therefore$ The roots are $2, 5, -2$